SUMMATION OF RECIPROCALS WHICH INVOLVE PRODUCTS OF TERMS FROM GENERALIZED FIBONACCI SEQUENCES–PART II

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1. INTRODUCTION

We consider the sequence $\{W_n\}$ defined, for all integers n, by

$$W_n = pW_{n-1} + W_{n-2},\ W_0 = a, W_1 = b. \tag{1.1}$$

Here a, b, and p are real numbers with $p \neq 0$. Write $\Delta = p^2 + 4$. Then it is known [3] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $\alpha = (p+\sqrt{\Delta})/2$, $\beta = (p-\sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$. As in [3], we will put $e_W = AB = b^2 - pab - a^2$.

We define a companion sequence $\{\overline{W}_n\}$ of $\{W_n\}$ by

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.3}$$

Aspects of this sequence have been treated, for example, in [2] and [4].

For $(W_0, W_1) = (0, 1)$, we write $\{W_n\} = \{U_n\}$ and, for $(W_0, W_1) = (2, p)$, we write $\{W_n\} = \{V_n\}$. The sequences $\{U_n\}$ and $\{V_n\}$ are generalizations of the Fibonacci and Lucas sequences, respectively. From (1.2) and (1.3) we see that $\overline{U}_n = V_n$ and $\overline{V}_n = \Delta U_n$. Thus, it is clear that $e_U = 1$ and $e_V = -\Delta = -(\alpha-\beta)^2$.

The purpose of this paper is to investigate the infinite sums

$$S_{k,m} = \sum_{n=1}^{\infty} \frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \tag{1.4}$$

and

$$T_{k,m} = \sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}W_{k(n+2m)}W_{k(n+3m)}}, \tag{1.5}$$

where k and m are positive integers with k even. Indeed, $S_{k,m}$ and the alternating sum derived from $T_{k,m}$ have been studied in [5], where k and m were assumed to be odd positive integers. Both sums were expressed in terms of an infinite sum, and certain finite sums. Here, however, with the altered constraints on k and m, we express $S_{k,m}$ and $T_{k,m}$ in terms of finite sums only.

Now, if $p > 0$, then $\alpha > 1$ and $\alpha > |\beta|$, so that

$$W_n \cong \frac{A}{\alpha-\beta}\alpha^n \quad \text{and} \quad \overline{W}_n \cong A\alpha^n. \tag{1.6}$$

On the other hand, if $p < 0$, then $\beta < -1$ and $|\beta| > |\alpha|$, and so

$$W_n \cong \frac{-B}{\alpha-\beta}\beta^n \quad \text{and} \quad \overline{W}_n \cong B\beta^n. \tag{1.7}$$

Hence, assuming that a and b are chosen so that no denominator vanishes, we see from the ratio test that $S_{k,m}$ and $T_{k,m}$ are absolutely convergent.

2. PRELIMINARY RESULTS

We require the following, in which k and m are taken to be integers with k *even*.

$$\frac{\beta^{kn}}{W_{kn}} - \frac{\beta^{k(n+m)}}{W_{k(n+m)}} = \frac{AU_{km}}{W_{kn}W_{k(n+m)}}, \tag{2.1}$$

$$W_{k(n+m)}W_{k(n+2m)} - W_{kn}W_{k(n+3m)} = e_W U_{km} U_{2km}, \tag{2.2}$$

$$W_{n+k} - W_{n-k} = \overline{W}_n U_k, \tag{2.3}$$

$$B\beta^n = W_{n+1} - \alpha W_n. \tag{2.4}$$

Identities (2.1) and (2.2) are readily proved with the use of (1.2) and (1.3). Identity (2.3) is a special case of (75) in [2], while (2.4) can be obtained from (3.2) in [1].

We will also make use of the following lemma.

Lemma 1: Let k and m be positive integers with k even. Then

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{e_W U_{km}} \left[\sum_{n=1}^{m} \frac{W_{kn+1}}{W_{kn}} - m\alpha \right]. \tag{2.5}$$

Proof: If we sum both sides of (2.1), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{AU_{km}} \sum_{n=1}^{m} \frac{\beta^{kn}}{W_{kn}},$$

and (2.5) follows from (2.4) and the fact that $e_W = AB$. □

In fact, under the hypotheses of Lemma 1, Theorem 2′ of [1] yields

$$\sum_{n=1}^{\infty} \frac{1}{W_{kn}W_{k(n+m)}} = \frac{1}{e_W U_k U_{km}} \left[\sum_{n=1}^{m} \frac{W_{k(n+1)}}{W_{kn}} - m\alpha^k \right]. \tag{2.6}$$

To see that (2.6) reduces to (2.5), we use the identities $\alpha^k = U_k\alpha + U_{k-1}$ and $W_{k(n+1)} = U_k W_{kn+1} + U_{k-1}W_{kn}$. From the first of these, which is easily proved by induction, we obtain the second if we first note that $\alpha^{kn+k} = U_k\alpha^{kn+1} + U_{k-1}\alpha^{kn}$, and write down the corresponding result involving β.

3. THE MAIN RESULTS

Our main results can now be given in two theorems.

Theorem 1: Let k and m be positive integers with k even. Then

$$S_{k,m} = \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{W_{kn}W_{k(n+m)}}. \tag{3.1}$$

Proof: Consider the expression

$$\frac{\beta^{kn}}{W_{kn}}-\frac{\beta^{k(n+m)}}{W_{k(n+m)}}+\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}. \tag{3.2}$$

Using (2.1), we can write this as

$$\frac{AU_{km}}{W_{kn}W_{k(n+m)}}+\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}} \tag{3.3}$$

or as

$$\frac{\beta^{kn}}{W_{kn}}-\left[\frac{\beta^{k(n+m)}}{W_{k(n+m)}}-\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}\right]=\frac{\beta^{kn}}{W_{kn}}-\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}. \tag{3.4}$$

Now

$$\begin{aligned}\frac{AU_{km}}{W_{kn}W_{k(n+m)}}-\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}&=\frac{AU_{km}}{W_{k(n+m)}}\left[\frac{1}{W_{kn}}-\frac{1}{W_{k(n+2m)}}\right]\\&=\frac{AU_{km}}{W_{k(n+m)}}\left[\frac{W_{k(n+2m)}-W_{kn}}{W_{kn}W_{k(n+2m)}}\right]\\&=\frac{AU_{km}^2\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},\quad\text{by (2.3).}\end{aligned} \tag{3.5}$$

But from (3.2)-(3.4), we then have

$$2\left[\frac{\beta^{kn}}{W_{kn}}-\frac{\beta^{k(n+m)}}{W_{k(n+m)}}+\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}\right]=\frac{\beta^{kn}}{W_{kn}}+\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}+\frac{AU_{km}^2\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},$$

so that

$$\frac{AU_{km}^2\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}=\left[\frac{\beta^{kn}}{W_{kn}}-\frac{\beta^{k(n+m)}}{W_{k(n+m)}}\right]-\left[\frac{\beta^{k(n+m)}}{W_{k(n+m)}}-\frac{\beta^{k(n+2m)}}{W_{k(n+2m)}}\right]. \tag{3.6}$$

Finally, summing both sides of (3.6), we obtain

$$AU_{km}^2S_{k,m}=\sum_{n=1}^{m}\frac{\beta^{kn}}{W_{kn}}-\sum_{n=1}^{m}\frac{\beta^{k(n+m)}}{W_{k(n+m)}},$$

and (3.1) follows from (2.1). □

If we put $W_n=F_n$ and $W_n=L_n$, and take $k=2$ and $m=1$, (3.1) becomes, respectively,

$$\sum_{n=1}^{\infty}\frac{L_{2n+2}}{F_{2n}F_{2n+2}F_{2n+4}}=\frac{1}{3}, \tag{3.7}$$

and

$$\sum_{n=1}^{\infty}\frac{F_{2n+2}}{L_{2n}L_{2n+2}L_{2n+4}}=\frac{1}{105}. \tag{3.8}$$

Theorem 2: Let k and m be positive integers with k even. Then

$$e_WU_{km}U_{2km}T_{k,m}=\frac{1}{e_W}\left[\frac{1}{U_{3km}}\sum_{n=1}^{3m}\frac{W_{kn+1}}{W_{kn}}-\frac{1}{U_{km}}\sum_{n=1}^{m}\frac{W_{kn+1}}{W_{kn}}\right]+\sum_{n=1}^{m}\frac{1}{W_{kn}W_{k(n+m)}}+\frac{m\alpha}{e_W}\left[\frac{1}{U_{km}}-\frac{3}{U_{3km}}\right]. \tag{3.9}$$

Proof: From (2.2), we see that

$$\frac{e_W U_{km} U_{2km}}{W_{kn} W_{k(n+m)} W_{k(n+2m)} W_{k(n+3m)}} = \frac{1}{W_{kn} W_{k(n+3m)}} - \frac{1}{W_{k(n+m)} W_{k(n+2m)}}.$$

Summing both sides we obtain, with the aid of (2.5),

$$e_W U_{km} U_{2km} T_{k,m} = \frac{1}{e_W U_{3km}}\left[\sum_{n=1}^{3m} \frac{W_{kn+1}}{W_{kn}} - 3m\alpha\right] - \left[\frac{1}{e_W U_{km}}\left[\sum_{n=1}^{m} \frac{W_{kn+1}}{W_{kn}} - m\alpha\right] - \sum_{n=1}^{m} \frac{1}{W_{kn} W_{k(n+m)}}\right],$$

which is (3.9). □

If we put $W_n = F_n$ and $W_n = L_n$, and take $k = 2$ and $m = 1$, (3.9) becomes, respectively,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+2} F_{2n+4} F_{2n+6}} = \frac{60\sqrt{5} - 133}{576}, \tag{3.10}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n} L_{2n+2} L_{2n+4} L_{2n+6}} = \frac{9\sqrt{5} - 20}{2160}. \tag{3.11}$$

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