

## CONTINUED FRACTIONS AND NEWTON'S APPROXIMATIONS, II

**Takao Komatsu**

Faculty of Education, Mie University, Tsu, Mie 514-8507 Japan

e-mail: komatsu@edu.mie-u.ac.jp

(Submitted July 1999)

In [2], Rieger showed a relationship between the golden section,  $g = (\sqrt{5} - 1)/2$ , and Newton approximation. In other words, he constructed a function so that every trial value in Newton approximation coincides with the even convergent of continued fraction expansion of  $g$ . In this note we give a more general result.

As usual,  $\theta = [a_0; a_1, a_2, \dots]$  denotes the continued fraction expansion of  $\theta$ , where

$$\begin{aligned} \theta &= a_0 + \theta_0, & a_0 &= \lfloor \theta \rfloor, \\ 1/\theta_{n-1} &= a_n + \theta_n, & a_n &= \lfloor 1/\theta_{n-1} \rfloor \quad (n = 1, 2, \dots). \end{aligned}$$

The  $k^{\text{th}}$  convergent  $p_k/q_k = [a_0; a_1, \dots, a_k]$  of  $\theta$  is then given by the recurrence relations

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \quad (k = 0, 1, \dots), & p_{-2} &= 0, & p_{-1} &= 1, \\ q_k &= a_k q_{k-1} + q_{k-2} \quad (k = 0, 1, \dots), & q_{-2} &= 1, & q_{-1} &= 0. \end{aligned}$$

Let  $a$  and  $b$  be positive integers and  $D = ab(ab + 4)$ . Set

$$\theta = [0; a, b, a, b, \dots] = [0; \overline{a, b}] = (\sqrt{D} - ab) / (2a),$$

satisfying  $a\theta^2 + ab\theta = b$ . Then  $\theta = \theta_2 = \theta_4 = \dots$  and

$$\theta_1 = \theta_3 = \theta_5 = \dots = [0; \overline{b, a}] = (\sqrt{D} - ab) / (2b).$$

Also, set

$$\hat{\theta} = (\sqrt{D} + ab) / (2a) = \theta + b = \theta_1^{-1} \quad \text{and} \quad \hat{\theta}_1 = (\sqrt{D} + ab) / (2b) = \theta_1 + a = \theta^{-1}.$$

Notice that  $\theta + \hat{\theta} = \sqrt{D}/a$ ,  $\theta_1 + \hat{\theta}_1 = \sqrt{D}/b$ ,  $\theta\hat{\theta} = b/a$ , and  $\theta_1\hat{\theta}_1 = a/b$ .

The arbitrary function  $H : [0, g] \rightarrow \mathbb{R}$  of class  $C^2$  may satisfy  $H(0) = 1$ ,  $H(g) = 0$ ,  $H'(x) < 0$ ,  $H''(x) > 0$  ( $0 \leq x < g$ ). Let

$$N(x) = x - \frac{H(x)}{H'(x)}.$$

Then Newton approximation applies with

$$x_0 = 0, \quad x_{n+1} = N(x_n) > x_n \quad (n = 0, 1, 2, \dots), \quad \lim_{x \rightarrow \infty} x_n = \theta.$$

We shall give  $H$  explicitly to show the following.

**Theorem:**  $x_n = \frac{p_{2n}}{q_{2n}}$  ( $n = 0, 1, 2, \dots$ ).

If we put  $a = b = 1$ , this is exactly the same as Rieger's case. It is clear that  $x_0 = 0 = p_0/q_0$ . Because  $ap_{2n} = bq_{2n-1}$  and  $p_{2n+1} = q_{2n}$  ( $n = 0, 1, 2, \dots$ ),

$$\frac{p_{2n+2}}{q_{2n+2}} = \frac{bp_{2n+1} + p_{2n}}{bq_{2n+1} + q_{2n}} = \frac{bq_{2n} + p_{2n}}{b(aq_{2n} + q_{2n-1}) + q_{2n}} = \frac{b + \frac{p_{2n}}{q_{2n}}}{ab + 1 + a \frac{p_{2n}}{q_{2n}}}$$

and

$$\frac{p_{2n+3}}{q_{2n+3}} = \frac{ap_{2n+2} + p_{2n+1}}{aq_{2n+2} + q_{2n+1}} = \frac{bq_{2n+1} + p_{2n+1}}{a(bq_{2n+1} + q_{2n}) + q_{2n+1}} = \frac{b + \frac{p_{2n+1}}{q_{2n+1}}}{ab + 1 + a \frac{p_{2n+1}}{q_{2n+1}}}$$

Thus, we set

$$N(x) = \frac{b+x}{ab+1+ax}$$

so Newton approximation applies with  $x_{n+1} = N(x_n)$  ( $n = 0, 1, 2, \dots$ ),  $\lim_{n \rightarrow \infty} x_n = \theta$ .  $y = N(x)$  is a hyperbola with asymptotes  $x = -(ab+1)/a$ ,  $y = 1/a$ ;  $N(\theta) = \theta$ ,  $N'(x) = 1/(ab+1+ax)^2 > 0$ . We take

$$D(x) = N(x) - x = \frac{b-ax-x^2}{ab+1+ax} = \frac{a(\theta-x)(\hat{\theta}+x)}{ab+1+ax} = \frac{b(1+\theta_1x)(1-\hat{\theta}_1x)}{ab+1+ax}$$

$y = D(x)$  is a hyperbola with asymptotes  $x = -(ab+1)/a$ ,  $x+y = 1/a$ ;

$$D(-\hat{\theta}) = D(\theta) = 0, \quad D(0) = \frac{b}{ab+1}, \quad D(x) > 0 \quad (-\hat{\theta} < x < \theta).$$

Since

$$\frac{\sqrt{D}}{D(x)} = \frac{(ab+1)\theta_1 - a}{1+\theta_1x} + \frac{(ab+1)\hat{\theta}_1 + a}{1-\hat{\theta}_1x},$$

we can choose

$$\begin{aligned} H(x) &= \exp\left(-\int_0^x \frac{dt}{D(t)}\right) \\ &= (1+\theta_1x)^{(ab+1-a\hat{\theta})/\sqrt{D}} (1-\hat{\theta}_1x)^{(ab+1+a\theta)/\sqrt{D}} \quad (0 \leq x \leq \theta) \end{aligned}$$

so that

$$\frac{H'(x)}{H(x)} = -\frac{1}{D(x)} \quad (0 \leq x < \theta).$$

We see that  $H(x) > 0$ ,  $H'(x) < 0$  ( $0 \leq x < g$ ),  $H(g) = 0$ , and  $H'(g) = 0$ . It follows that  $H''(x) > 0$  ( $0 \leq x < g$ ). We also note that

$$x_0 = 0, \quad x_{n+1} = \frac{b+x_n}{ab+1+ax_n} \quad (n = 0, 1, 2, \dots).$$

Of course,  $N(x)$  keeps the property of mediants. Let integers  $\alpha, \beta, \gamma$ , and  $\delta$  be  $\beta > 0$ ,  $\delta > 0$ ,  $\beta\gamma - \alpha\delta = 1$ , then  $(\alpha, \beta) = (\gamma, \delta) = 1$ , and

$$\frac{\alpha}{\beta} < \frac{\alpha+\gamma}{\beta+\delta} < \frac{\gamma}{\delta}.$$

Let  $\alpha' = b\beta + \alpha$ ,  $\beta' = (ab+1)\beta + a\alpha > 0$ ,  $\gamma' = b\delta + \gamma$ , and  $\delta' = (ab+1)\delta + a\gamma > 0$ . Then

$$(\alpha', \beta') = (b\beta + \alpha, (ab+1)\beta + a\alpha) = (b\beta + \alpha, \beta) = (\alpha, \beta) = 1, \quad (\gamma', \delta') = 1,$$

$$N\left(\frac{\alpha}{\beta}\right) = \frac{\alpha'}{\beta'}, \quad N\left(\frac{\alpha+\gamma}{\beta+\delta}\right) = \frac{\alpha'+\gamma'}{\beta'+\delta'}, \quad N\left(\frac{\gamma}{\delta}\right) = \frac{\gamma'}{\delta'}.$$

**Remark:** If we set  $x_0 = 1/a$  as the initial value, then

$$x_{n+1} = N(x_n) < x_n \quad (n = 0, 1, 2, \dots), \quad \lim_{y \rightarrow \infty} x_n = \theta,$$

and  $x_n = p_{2n+1}/q_{2n+1}$  ( $n = 0, 1, 2, \dots$ ). However, the corresponding  $H(x)$  does not exist for  $x > \theta$ . Indeed,

$$\frac{p_1}{q_1} = \frac{1}{a} > \frac{p_3}{q_3} > \dots > \frac{p_{2n+1}}{q_{2n+1}} > \dots > \theta.$$

Further generalization seems nearly impossible. For example, if  $\theta = [0; \overline{a, b, c, d}]$ ,  $p_{4n+4}/q_{4n+4}$  cannot be expressed by the linear relation of  $p_{4n}$  and  $q_{4n}$ . Hence, we cannot give  $N(x)$  as well as  $D(x)$  and  $H(x)$ .

A different aspect of this topic can be seen in [1].

#### REFERENCES

1. T. Komatsu. "Continued Fractions and Newton Approximations." *Math. Communications* 4 (1999):167-76.
2. G. J. Rieger. "The Golden Section and Newton Approximation." *The Fibonacci Quarterly* 37.2 (1999):178-79.

AMS Classification Numbers: 11A55, 11B39, 41A05



### Author and Title Index

The TITLE, AUTHOR, ELEMENTARY PROBLEMS, ADVANCED PROBLEMS, and KEY-WORD indices for Volumes 1-38.3 (1963-July 2000) of *The Fibonacci Quarterly* have been completed by Dr. Charles K. Cook. It is planned that the indices will be available on The Fibonacci Web Page. Anyone wanting their own disc copy should send two 1.44 MB discs and a self-addressed stamped envelope with enough postage for two discs. PLEASE INDICATE WORDPERFECT 6.1 OR MS WORD 97.

Send your request to:

PROFESSOR CHARLES K. COOK  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF SOUTH CAROLINA AT SUMTER  
1 LOUISE CIRCLE  
SUMTER, SC 29150-2498