CONTINUED FRACTIONS AND NEWTON'S APPROXIMATIONS, II

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In [2], Rieger showed a relationship between the golden section, $g = (\sqrt{5}-1)/2$, and Newton approximation. In other words, he constructed a function so that every trial value in Newton approximation coincides with the even convergent of continued fraction expansion of g. In this note we give a more general result.

As usual, $\theta = [a_0; a_1, a_2, \dots]$ denotes the continued fraction expansion of θ, where

$$\theta = a_0 + \theta_0, \qquad a_0 = \lfloor \theta \rfloor,$$
$$1/\theta_{n-1} = a_n + \theta_n, \qquad a_n = \lfloor 1/\theta_{n-1} \rfloor \quad (n = 1, 2, \dots).$$

The k^{th} convergent $p_k/q_k = [a_0; a_1, \dots, a_k]$ of θ is then given by the recurrence relations

$$p_k = a_k p_{k-1} + p_{k-2} \quad (k = 0, 1, \dots), \quad p_{-2} = 0, \quad p_{-1} = 1,$$
$$q_k = a_k q_{k-1} + q_{k-2} \quad (k = 0, 1, \dots), \quad q_{-2} = 1, \quad q_{-1} = 0.$$

Let a and b be positive integers and $D = ab(ab+4)$. Set

$$\theta = [0; a, b, a, b, \dots] = [0; \overline{a, b}] = (\sqrt{D} - ab)/(2a),$$

satisfying $a\theta^2 + ab\theta = b$. Then $\theta = \theta_2 = \theta_4 = \cdots$ and

$$\theta_1 = \theta_3 = \theta_5 = \cdots = [0; \overline{b, a}] = (\sqrt{D} - ab)/(2b).$$

Also, set

$$\hat{\theta} = (\sqrt{D} + ab)/(2a) = \theta + b = \theta_1^{-1} \quad \text{and} \quad \hat{\theta}_1 = (\sqrt{D} + ab)/(2b) = \theta_1 + a = \theta^{-1}.$$

Notice that $\theta + \hat{\theta} = \sqrt{D}/a$, $\theta_1 + \hat{\theta}_1 = \sqrt{D}/b$, $\theta\hat{\theta} = b/a$, and $\theta_1\hat{\theta}_1 = a/b$.

The arbitrary function $H : [0, g] \to \mathbb{R}$ of class C^2 may satisfy $H(0) = 1$, $H(g) = 0$, $H'(x) < 0$, $H''(x) > 0$ $(0 \le x < g)$. Let

$$N(x) = x - \frac{H(x)}{H'(x)}.$$

Then Newton approximation applies with

$$x_0 = 0, \quad x_{n+1} = N(x_n) > x_n \quad (n = 0, 1, 2, \dots), \quad \lim_{x \to \infty} x_n = \theta.$$

We shall give H explicitly to show the following.

Theorem: $x_n = \dfrac{p_{2n}}{q_{2n}}$ $(n = 0, 1, 2, \dots)$.

If we put $a = b = 1$, this is exactly the same as Rieger's case. It is clear that $x_0 = 0 = p_0/q_0$. Because $ap_{2n} = bq_{2n-1}$ and $p_{2n+1} = q_{2n}$ $(n = 0, 1, 2, \dots)$,

$$\frac{p_{2n+2}}{q_{2n+2}}=\frac{bp_{2n+1}+p_{2n}}{bq_{2n+1}+q_{2n}}=\frac{bq_{2n}+p_{2n}}{b(aq_{2n}+q_{2n-1})+q_{2n}}=\frac{b+\frac{p_{2n}}{q_{2n}}}{ab+1+a\frac{p_{2n}}{q_{2n}}}$$

and

$$\frac{p_{2n+3}}{q_{2n+3}}=\frac{ap_{2n+2}+p_{2n+1}}{aq_{2n+2}+q_{2n+1}}=\frac{bq_{2n+1}+p_{2n+1}}{a(bq_{2n+1}+q_{2n})+q_{2n+1}}=\frac{b+\frac{p_{2n+1}}{q_{2n+1}}}{ab+1+a\frac{p_{2n+1}}{q_{2n+1}}}.$$

Thus, we set

$$N(x)=\frac{b+x}{ab+1+ax}$$

so Newton approximation applies with $x_{n+1}=N(x_n)$ $(n=0,1,2,\dots)$, $\lim_{n\to\infty}x_n=\theta$. $y=N(x)$ is a hyperbola with asymptotes $x=-(ab+1)/a$, $y=1/a$; $N(\theta)=\theta$, $N'(x)=1/(ab+1+ax)^2>0$. We take

$$D(x)=N(x)-x=\frac{b-abx-ax^2}{ab+1+ax}=\frac{a(\theta-x)(\hat{\theta}+x)}{ab+1+ax}=\frac{b(1+\theta_1x)(1-\hat{\theta}_1x)}{ab+1+ax}.$$

$y=D(x)$ is a hyperbola with asymptotes $x=-(ab+1)/a$, $x+y=1/a$;

$$D(-\hat{\theta})=D(\theta)=0,\ \ D(0)=\frac{b}{ab+1},\ \ D(x)>0\ \ (-\hat{\theta}<x<\theta).$$

Since

$$\frac{\sqrt{D}}{D(x)}=\frac{(ab+1)\theta_1-a}{1+\theta_1x}+\frac{(ab+1)\hat{\theta}_1+a}{1-\hat{\theta}_1x},$$

we can choose

$$\begin{aligned}H(x)&=\exp\left(-\int_0^x\frac{dt}{D(t)}\right)\\&=(1+\theta_1x)^{(ab+1-a\hat{\theta})/\sqrt{D}}(1-\hat{\theta}_1x)^{(ab+1+a\theta)/\sqrt{D}}\quad(0\le x\le\theta)\end{aligned}$$

so that

$$\frac{H'(x)}{H(x)}=-\frac{1}{D(x)}\quad(0\le x<\theta).$$

We see that $H(x)>0$, $H'(x)<0$ $(0\le x<g)$, $H(g)=0$, and $H'(g)=0$. It follows that $H''(x)>0$ $(0\le x<g)$. We also note that

$$x_0=0,\ \ x_{n+1}=\frac{b+x_n}{ab+1+ax_n}\quad(n=0,1,2,\dots).$$

Of course, $N(x)$ keeps the property of mediants. Let integers α,β,γ, and δ be $\beta>0$, $\delta>0$, $\beta\gamma-\alpha\delta=1$, then $(\alpha,\beta)=(\gamma,\delta)=1$, and

$$\frac{\alpha}{\beta}<\frac{\alpha+\gamma}{\beta+\delta}<\frac{\gamma}{\delta}.$$

Let $\alpha'=b\beta+\alpha$, $\beta'=(ab+1)\beta+a\alpha>0$, $\gamma'=b\delta+\gamma$, and $\delta'=(ab+1)\delta+a\gamma>0$. Then

$$(\alpha',\beta')=(b\beta+\alpha,(ab+1)\beta+a\alpha)=(b\beta+\alpha,\beta)=(\alpha,\beta)=1,\ \ (\gamma',\delta')=1,$$

$$N\left(\frac{\alpha}{\beta}\right)=\frac{\alpha'}{\beta'},\quad N\left(\frac{\alpha+\gamma}{\beta+\delta}\right)=\frac{\alpha'+\gamma'}{\beta'+\delta'},\quad N\left(\frac{\gamma}{\delta}\right)=\frac{\gamma'}{\delta'}.$$

Remark: If we set $x_0 = 1/a$ as the initial value, then

$$x_{n+1}=N(x_n)<x_n \quad (n=0,1,2,\dots),\quad \lim_{y\to\infty} x_n=\theta,$$

and $x_n = p_{2n+1}/q_{2n+1}$ $(n=0,1,2,\dots)$. However, the corresponding $H(x)$ does not exist for $x>\theta$. Indeed,

$$\frac{p_1}{q_1}=\frac{1}{a}>\frac{p_3}{q_3}>\cdots>\frac{p_{2n+1}}{q_{2n+1}}>\cdots>\theta.$$

Further generalization seems nearly impossible. For example, if $\theta=[0;\overline{a,b,c,d}]$, p_{4n+4}/q_{4n+4} cannot be expressed by the linear relation of p_{4n} and q_{4n}. Hence, we cannot give $N(x)$ as well as $D(x)$ and $H(x)$.

A different aspect of this topic can be seen in [1].

REFERENCES

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