INVARIANT SEQUENCES UNDER BINOMIAL TRANSFORMATION

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1. INTRODUCTION

The classical binomial inversion formula states that $a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k b_k$ $(n = 0, 1, 2, \ldots)$ if and only if $b_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k$ $(n = 0, 1, 2, \ldots)$. In this paper we study those sequences $\{a_n\}$ such that $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = \pm a_n$ $(n = 0, 1, 2, \ldots)$. If $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = a_n$ $(n \ge 0)$, we say that $\{a_n\}$ is an invariant sequence. If $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = -a_n$ $(n \ge 0)$, we say that $\{a_n\}$ is an inverse invariant sequence.

Throughout this paper, let *IS* denote the set of invariant sequences, and let *IIS* denote the set of inverse invariant sequences. We mention that it can be proved easily that $\{a_n\} \in IIS$ if and only if $a_0 = 0$ and $\left\{\frac{a_{n+1}}{n+1}\right\} \in IS$ or $\{na_{n-1}\} \in IS$.

In Section 2 we list some typical examples of invariant sequences. For example,

$$\left\{\frac{1}{2^n}\right\},\ \left\{\frac{1}{\binom{n+2m-1}{m}}\right\},\ \left\{(-1)^n\int_0^{-1}\binom{x}{n}dx\right\},\ \{nF_{n-1}\},\ \{L_n\},\ \{(-1)^n B_n\} \in IS,$$

where $\{F_n\}$, $\{L_n\}$, and $\{B_n\}$ denote the Fibonacci sequence, Lucas sequence, and Bernoulli numbers, respectively. The Bernoulli numbers $\{B_n\}$ are given by $B_0 = 1$ and $\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$ $(n \ge 2)$.

In Section 3 we investigate the generating functions of invariant sequences. As a consequence, it is proved that $\{a_n\} \in IS$ if and only if there is a sequence $\{\alpha_{2k}\}$ such that

$$a_n = \frac{1}{2^n}\sum_{\substack{k=0\\ 2|k}}^{n}\binom{n}{k}\alpha_k \quad (n = 0, 1, 2, \ldots).$$

Section 4 is devoted to recursion relations for invariant sequences. The main result is

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k) - (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)A_{n-k} = 0 \quad (n = 0, 1, 2, \ldots),$$

where $\{A_n\} \in IS$ and f is an arbitrary function. We also point out similar recursion relations for inverse invariant sequences. As consequences, if $\{B_n\}$, $\{F_n\}$, and $\{L_n\}$ denote the Bernoulli numbers, Fibonacci sequence, and Lucas sequence, respectively, then

$$\sum_{k=0}^{n}\binom{n}{k}\left((-1)^{n-k}f(k) - \sum_{s=0}^{k}\binom{k}{s}f(s)\right)B_{n-k} = 0 \quad (n = 0, 1, 2, \ldots),$$

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k) + (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)F_{n-k} = 0 \quad (n = 0, 1, 2, \ldots),$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k) - (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)L_{n-k} = 0 \quad (n = 0, 1, 2, \ldots).$$

This gives infinitely many recursion relations for the Bernoulli numbers, Fibonacci sequence, and Lucas sequence.

In Section 5 we establish the following transformation formulas:

(1.1) Let $\{F_n\}$ be the Fibonacci sequence. If $a_n = \sum_{k=0}^{n} F_{k-1} b_{n-k}$ $(n = 0, 1, 2, \ldots)$ then $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

(1.2) Let $\{F_n\}$ be the Fibonacci sequence. If $\sum_{k=0}^{n} a_k b_{n-k} = F_{n+1}$ $(n = 0, 1, 2, \ldots)$, then $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

(1.3) Let $\{a_n\}$ and $\{A_n\}$ be two sequences satisfying $\sum_{k=0}^{n} a_{n-k} A_k = 1$ $(n = 0, 1, 2, \ldots)$. Then $\{a_n\} \in IS$ if and only if $\{A_n\} \in IS$.

(1.4) Let $\{a_n\}$ and $\{A_n\}$ be two sequences satisfying $\sum_{k=0}^{n} \binom{n}{k} a_{n-k} A_k = 1$ $(n = 0, 1, 2, \ldots)$. Then $\{a_n\} \in IS$ if and only if $\{A_n\} \in IS$.

(1.5) If $\{A_n\} \in IS$ with $A_0 \neq 0$ and $\{a_n\}$ is given by $a_0 A_0 = 1$ and $\sum_{k=0}^{n} a_{n-k} A_k = 0$ $(n = 1, 2, 3, \ldots)$, then $\{a_{n+2}\} \in IS$ and $\{\sum_{k=0}^{n} a_k\} \in IS$.

(1.6) If $\{A_n\} \in IS$ with $A_0 \neq 0$ and $\{a_n\}$ is given by $a_0 A_0 = 2$ and $\sum_{k=0}^{n} a_{n-k} A_k = 1$ $(n = 1, 2, 3, \ldots)$, then $\{a_{n+1}\} \in IIS$ and $\{na_n\} \in IS$.

(1.7) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonzero sequences satisfying $c_n = \frac{1}{n+1} \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0, 1, 2, \ldots)$. If two of the three sequences are invariant sequences, then the other sequence is also an invariant sequence.

(1.8) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonzero sequences satisfying $c_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ $(n = 0, 1, 2, \ldots)$. If two of the three sequences are invariant sequences, then the other sequence is also an invariant sequence.

2. EXAMPLES OF INVARIANT SEQUENCES

In this section we present some typical examples of invariant sequences. One can easily verify the following examples:

Example 1: $\{1/2^n\} \in IS$.

Example 2: If $A_0 = 2$ and $A_n = 1$ $(n \geq 1)$, then $\{A_n\} \in IS$.

Example 3: If $A_0 = A_1 = 0$ and $A_n = n$ $(n \geq 2)$, then $\{A_n\} \in IS$.

Example 4: If $v_0(t) = 2$, $v_1(t) = 1$, $v_{n+1}(t) = v_n(t) + tv_{n-1}(t)$ $(n \geq 1)$, then $\{v_n(t)\} \in IS$.

Example 5: If $u_0(t) = 0$, $u_1(t) = 1$, $u_{n+1}(t) = u_n(t) + tu_{n-1}(t)$ $(n \geq 1)$, then $\{u_n(t)\} \in IIS$, $\{nu_{n-1}(t)\} \in IS$, and $\left\{\frac{u_{n+1}(t)}{n+1}\right\} \in IS$.

Example 6: If $T_n(x) = \cos(n \arccos x)$ is the n^{th} Tchebychev polynomial, then $\{T_n(x)/(2x)^n\} \in IS$.

Example 7: Let $\{B_n\}$ be the Bernoulli numbers. Then $\{(-1)^n B_n\} \in IS$ and $\{(-1)^{n+1}(2^{n+1}-1)B_{n+1}/(n+1)\} \in IS$.

For further examples, we need the following Vandermonde identity:

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n},$$

where x and y are real numbers and n is a nonnegative integer.

Example 8: If $x \neq 0, 1, 2, \ldots$, then $\{\binom{x/2}{n}/\binom{x}{n}\} \in IS$.

By Vandermonde's identity, it is clear that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\binom{x/2}{k}}{\binom{x}{k}} = \frac{1}{\binom{x}{n}}\sum_{k=0}^{n}\binom{x}{n}\binom{n}{k}(-1)^k\frac{\binom{x/2}{k}}{\binom{x}{k}}$$

$$= \frac{1}{\binom{x}{n}}\sum_{k=0}^{n}\binom{x}{k}\binom{x-k}{n-k}(-1)^k\frac{\binom{x/2}{k}}{\binom{x}{k}}$$

$$= \frac{(-1)^n}{\binom{x}{n}}\sum_{k=0}^{n}\binom{n-x-1}{n-k}\binom{x/2}{k} = \frac{(-1)^n}{\binom{x}{n}}\binom{n-(x/2)-1}{n} = \frac{\binom{x/2}{k}}{\binom{x}{n}}.$$

Example 9: If $m \in \{1, 2, 3, \ldots\}$, then $\{1/\binom{n+2m-1}{m}\} \in IS$. Since

$$\frac{\binom{-m}{n}}{\binom{-2m}{n}} = \frac{(m+n-1)!}{(m-1)!}\cdot\frac{(2m-1)!}{(2m+n-1)!} = \frac{\binom{2m-1}{m}}{\binom{n+2m-1}{m}},$$

the result follows from Example 8 immediately.

Example 10: $\{\binom{2n}{n}/2^{2n}\} \in IS$.

Clearly, $\binom{-1/2}{n}/\binom{-1}{n} = \binom{2n}{n}/2^{2n}$. So the example is a special case of Example 8.

Example 11: If $m \in \{0, 1, 2, \ldots\}$, then

$$\left\{(-1)^n\int_0^{2m-1}\binom{x}{n+2m}dx\right\} \in IS.$$

By Vandermonde's identity,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+2m} = \sum_{r=0}^{n}\binom{n}{n-r}\binom{x}{n+2m-r} = \sum_{r=0}^{n+2m}\binom{n}{r}\binom{x}{n+2m-r} = \binom{n+x}{n+2m}.$$

Set

$$A_n(x) = \binom{n+x}{n+2m} + (-1)^n\binom{x}{n+2m}.$$

Then $\{A_n(x)\} \in IS$ by the above and the binomial inversion formula. Note that

$$\int_0^{2m-1}\binom{n+x}{n+2m}dx = \int_0^{2m-1}\binom{n+2m-1-x}{n+2m}dx = (-1)^n\int_0^{2m-1}\binom{x}{n+2m}dx.$$

So we have

$$(-1)^n\int_0^{2m-1}\binom{x}{n+2m}dx = \frac{1}{2}\int_0^{2m-1}A_n(x)\,dx$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\cdot\frac{1}{2}\int_0^{2m-1}A_k(x)\,dx = \sum_{k=0}^{n}\binom{n}{k}\int_0^{2m-1}\binom{x}{k+2m}dx.$$

3. THE GENERATING FUNCTIONS OF INVARIANT SEQUENCES

For any sequence $\{a_n\}$ the formal power series $\sum_{n=0}^{\infty}a_nx^n$ is called the generating function of $\{a_n\}$, and the formal power series $\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}$ is called the exponential generating function of $\{a_n\}$.

Theorem 3.1: Let $a(x)=\sum_{n=0}^{\infty}a_nx^n$. Then $\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k=\pm a_n$ $(n=0,1,2,\dots)$ if and only if $a(x)$ satisfies the equation $a(\frac{x}{x-1})=\pm(1-x)a(x)$.

Proof: Clearly,

$$\begin{aligned}(1-x)^{-1}a\left(\frac{x}{x-1}\right)&=\sum_{k=0}^{\infty}(-1)^k a_k x^k(1-x)^{-1-k}=\sum_{k=0}^{\infty}(-1)^k a_k x^k\sum_{r=0}^{\infty}\binom{-1-k}{r}(-x)^r\\&=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}(-1)^{n-r}a_{n-r}\binom{-1-(n-r)}{r}(-1)^r\right)x^n\\&=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}a_{n-r}\right)x^n=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}(-1)^r a_r\right)x^n.\end{aligned}\tag{3.1}$$

Therefore, the result follows.

Remark 3.1: Formula (3.1) is known (see [1]).

Corollary 3.1: Let $\{a_n\}$ be a given sequence. Then:

(a) $\{a_n\}\in IS$ if and only if $\{2a_{n+1}-a_n\}\in IIS$.

(b) $\{a_n\}\in IIS$ if and only if $a_0=0$ and $\{2a_{n+1}-a_n\}\in IS$.

(c) If $\{a_n\}\in IS$, then $\{a_{n+2}-a_{n+1}\}\in IS$.

(d) If $\{a_n\}\in IIS$, then $\{a_{n+2}-a_{n+1}\}\in IIS$.

Proof: Let $b_n=2a_{n+1}-a_n$, $b(x)=\sum_{n=0}^{\infty}b_nx^n$, and $a(x)=\sum_{n=0}^{\infty}a_nx^n$. It is clear that

$$b(x)=\frac{2(a(x)-a_0)}{x}-a(x)=\frac{2-x}{x}a(x)-\frac{2}{x}a_0,$$

and so

$$b\left(\frac{x}{x-1}\right)=\frac{x-2}{x}a\left(\frac{x}{x-1}\right)-\frac{2(x-1)}{x}a_0.$$

Thus,

$$\begin{aligned}a\left(\frac{x}{x-1}\right)=\pm(1-x)a(x)\Leftrightarrow b\left(\frac{x}{x-1}\right)&=\frac{\pm x-2}{x}(1-x)a(x)-\frac{2(x-1)}{x}a_0\\&=\pm(x-1)b(x)+\frac{2(x-1)}{x}(\pm a_0-a_0).\end{aligned}$$

This, together with Theorem 3.1, deduces that $\{a_n\}\in IS\Leftrightarrow\{b_n\}\in IIS$, $\{a_n\}\in IIS\Leftrightarrow a_0=0$, and $\{b_n\}\in IS$. Hence,

$$\begin{aligned}&\{a_n\}\in IS(IIS)\Rightarrow\{b_n\}\in IIS(IS)\Rightarrow\{2b_{n+1}-b_n\}\in IS(IIS)\\&\Rightarrow\{4(a_{n+2}-a_{n+1})+a_n\}\in IS(IIS)\Rightarrow\{a_{n+2}-a_{n+1}\}\in IS(IIS).\end{aligned}$$

This completes the proof.

Remark 3.2: If $a_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\alpha_k+\alpha_n$, then by the binomial inversion formula. Conversely, if $\{a_n\}\in IS$, we may take $\alpha_n=a_{n+1}$ by Corollary 3.1(a).

Corollary 3.2: Suppose $\{a_n\}\in IS$, $A_0=A_1=0$, and $A_n=\sum_{k=0}^{n-2}a_k$ $(n\ge 2)$. Then $\{A_n\}\in IS$.

Proof: Let $s_{-1}=s_{-2}=0$ and $s_n=\sum_{k=0}^{n}a_k$ $(n\ge 0)$. Then $A_n=s_{n-2}$ for $n\ge 0$. If $a(x)$ and $A(x)$ are the generating functions of $\{a_n\}$ and $\{A_n\}$, respectively, we see that

$$A(x)=x^2\sum_{n=0}^{\infty}s_nx^n=\frac{x^2}{1-x}a(x).$$

Hence,

$$A\left(\frac{x}{x-1}\right)=\frac{x^2}{1-x}(1-x)a(x)=(1-x)A(x).$$

This, together with Theorem 3.1, proves the corollary.

Theorem 3.2: Let $A^*(x)$ be the exponential generating function of $\{A_n\}$. Then $\{A_n\}\in IS$ if and only if $A^*(x)e^{-x/2}$ is an even function, and $\{A_n\}\in IIS$ if and only if $A^*(x)e^{-x/2}$ is an odd function.

Proof: Clearly

$$A^*(-x)e^x=\sum_{k=0}^{\infty}(-1)^kA_k\frac{x^k}{k!}\sum_{m=0}^{\infty}\frac{x^m}{m!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^kA_k\right)\frac{x^n}{n!}.$$

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kA_k=\pm A_n\quad(n=0,1,2,\dots)$$
$$\Leftrightarrow A^*(-x)e^x=\pm A^*(x)\Leftrightarrow A^*(-x)e^{x/2}=\pm A^*(x)e^{-x/2}.$$

This completes the proof.

Remark 3.3: The first part of Theorem 3.2 is due to Zhi-Wei Sun.

Corollary 3.3: Let $\{A_n\}$ be a given sequence. Then

(a) $\{A_n\}\in IS$ if and only if there exists a sequence $\{a_{2k}\}$ such that

$$A_n=\frac{1}{2^n}\sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}a_k\quad(n=0,1,2,\dots).$$

(b) $\{A_n\}\in IIS$ if and only if there exists a sequence $\{a_{2k+1}\}$ such that

$$A_n=\frac{1}{2^n}\sum_{\substack{k=0\\2\nmid k}}^{n}\binom{n}{k}a_k\quad(n=0,1,2,\dots).$$

Proof: Suppose

$$A^*(x)=\sum_{n=0}^{\infty}A_n\frac{x^n}{n!}\quad\text{and}\quad\alpha(x)=A^*(x)e^{-x/2}=\sum_{n=0}^{\infty}\alpha_n\frac{x^n}{n!}.$$

Then $A^*(x)=\alpha(x)e^{x/2}$, and hence

$$A_n=\sum_{k=0}^{n}\binom{n}{k}\alpha_k\frac{1}{2^{n-k}}\quad(n=0,1,2,\dots).$$

If $\{A_n\}\in IS$, then $\alpha(-x)=\alpha(x)$ by Theorem 3.2. Hence, $\alpha_{2n-1}=0$ for $n=1,2,3,\dots$. On setting $a_k=2^k\alpha_k$, we see that

$$A_n=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k=\frac{1}{2^n}\sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}a_k\quad(n=0,1,2,\dots).$$

Conversely, if there is a sequence $\{a_{2k}\}$ for which

$$A_n = \frac{1}{2^n}\sum_{\substack{k=0\\2|k}}^{n}\binom{n}{k}a_k \quad (n = 0, 1, 2, \ldots),$$

then

$$A_n = \sum_{k=0}^{n}\binom{n}{k}\alpha_k \frac{1}{2^{n-k}}$$

for

$$\alpha_k = \begin{cases} a_k/2^k & \text{if } 2|k, \\ 0 & \text{if } 2\nmid k. \end{cases}$$

So $A^*(x)e^{-x/2} = \alpha(x)$ is an even function. It then follows from Theorem 3.2. that $\{A_n\} \in IS$. This proves part (a). Part (b) can be proved similarly.

4. RECURSION RELATIONS FOR INVARIANT SEQUENCES

In this section we present infinitely many recursion relations for invariant sequences.

Theorem 4.1: Let $\{A_n\} \in IS$. For any function f, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k) - (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)A_{n-k} = 0 \quad (n = 0, 1, 2, \ldots).$$

Proof: Let $A^*(x)$ be the exponential generating function of $\{A_n\}$,

$$C_0^*(x) = \sum_{k=0}^{\infty}\left((-1)^k f(k) + \sum_{s=0}^{k}\binom{k}{s}f(s)\right)\frac{x^k}{k!}$$

and

$$C_1^*(x) = \sum_{k=0}^{\infty}\left((-1)^k f(k) - \sum_{s=0}^{k}\binom{k}{s}f(s)\right)\frac{x^k}{k!}.$$

From the binomial inversion formula, we know that

$$\left\{(-1)^k f(k) + \sum_{s=0}^{k}\binom{k}{s}f(s)\right\} \in IS \quad \text{and} \quad \left\{(-1)^k f(k) - \sum_{s=0}^{k}\binom{k}{s}f(s)\right\} \in IIS.$$

So, by Theorem 3.2, $C_0^*(x)e^{-x/2}$ is an even function and $C_1^*(x)e^{-x/2}$ is an odd function.

Now suppose

$$a_n = \sum_{k=0}^{n}\binom{n}{k}\left(f(k) - (-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)A_{n-k}.$$

If n is even, then

$$a_n = \sum_{k=0}^{n}\binom{n}{k}\left((-1)^k f(k) - \sum_{s=0}^{k}\binom{k}{s}f(s)\right)(-1)^{n-k}A_{n-k}.$$

So $a_n/n!$ is the coefficient of x^n in the power series expansion of $A^*(-x)C_1^*(x)$. Since $A^*(-x)\cdot C_1^*(x)$ $(= A^*(-x)e^{x/2}\cdot C_1^*(x)e^{-x/2})$ is an odd function by Theorem 3.2, we find $a_n = 0$ for all even n.

Similarly, when n is odd, $-a_n/n!$ is the coefficient of x^n in the power series expansion of $A^*(-x)C_0^*(x)$. Since $A^*(-x)C_0^*(x)$ $(= A^*(-x)e^{x/2}\cdot C_0^*(x)e^{-x/2})$ is an even function by Theorem 3.2, we must have $a_n = 0$ for all odd n. This concludes the proof.

Corollary 4.1: Let $\{B_n\}$ be the Bernoulli numbers. For any function f, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left((-1)^{n-k}f(k)-\sum_{s=0}^{k}\binom{k}{s}f(s)\right)B_{n-k}=0 \quad (n=0,1,2,\ldots).$$

Proof: This is immediate from Example 7 and Theorem 4.1.

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci sequence and Lucas sequence, respectively. It is easily seen that $\{F_n\}\in IIS$ and $\{L_n\}\in IS$. Thus, by Corollary 4.1, we have

$$\sum_{k=0}^{n}\binom{n}{k}F_kB_{n-k}=0 \quad (n=0,2,4,\ldots) \tag{4.1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}L_kB_{n-k}=0 \quad (n=1,3,5,\ldots). \tag{4.2}$$

This result has been given by the author in [2].

Corollary 4.2: Let $\{L_n\}$ be the Lucas sequence. For any function f, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k)-(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)L_{n-k}=0 \quad (n=0,1,2,\ldots).$$

Using the method in the proof of Theorem 4.1, one can similarly prove

Theorem 4.2: Let $\{A_n\}\in IIS$. For any function f, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k)+(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)A_{n-k}=0 \quad (n=0,1,2,\ldots).$$

Corollary 4.3: Let $\{F_n\}$ denote the Fibonacci sequence. For any function f, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(f(k)+(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}f(s)\right)F_{n-k}=0 \quad (n=0,1,2,\ldots).$$

5. TRANSFORMATION FORMULAS FOR INVARIANT SEQUENCES

Theorem 5.1: Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonzero sequences satisfying $c_n=\frac{1}{n+1}\sum_{k=0}^{n}a_kb_{n-k}$ $(n=0,1,2,\ldots)$. If two of the three sequences are invariant sequences, then the other sequence is also an invariant sequence.

Proof: Let $d_0=0$ and $d_{n+1}=(n+1)c_n$ $(n\geq 0)$. If $a(x)$, $b(x)$, and $d(x)$ are the generating functions of $\{a_n\}$, $\{b_n\}$, and $\{d_n\}$, respectively, then

$$d(x)=\sum_{n=0}^{\infty}(n+1)c_nx^{n+1}=xa(x)b(x).$$

Suppose $\{a_n\} \in IS$. Then $a(\frac{x}{x-1}) = (1-x)a(x)$. Since

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k c_k = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k d_{k+1} = -\frac{1}{n+1}\sum_{r=0}^{n+1}\binom{n+1}{r}(-1)^r d_r,$$

using Theorem 3.1, we see that

$$\begin{aligned}\{c_n\} \in IS \Leftrightarrow \{d_n\} \in IIS \Leftrightarrow d\left(\frac{x}{x-1}\right) = -(1-x)d(x)\\ \Leftrightarrow a\left(\frac{x}{x-1}\right)b\left(\frac{x}{x-1}\right) = (1-x)^2 a(x)b(x)\\ \Leftrightarrow b\left(\frac{x}{x-1}\right) = (1-x)b(x) \Leftrightarrow \{b_n\} \in IS.\end{aligned}$$

This is the result.

Corollary 5.1: Let $\{a_n\}$ and $\{b_n\}$ be two sequences for which

$$\sum_{k=0}^{n} a_k b_{n-k} = 1 \quad (n = 0, 1, 2, \ldots).$$

Then $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

Proof: Putting $c_n = \frac{1}{n+1}$ in Theorem 5.1 yields the result.

Corollary 5.2: Let $\{F_n\}$ be the Fibonacci sequence. Then, if $\{a_n\}$ and $\{b_n\}$ satisfy the relation $\sum_{k=0}^{n} a_k b_{n-k} = F_{n+1}$ $(n = 0, 1, 2, \ldots)$, we have $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

Proof: It is easy to check that $\left\{\frac{F_{n+1}}{n+1}\right\} \in IS$. This, together with Theorem 5.1, gives the result.

Theorem 5.2: Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonzero sequences satisfying

$$c_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k} \quad (n = 0, 1, 2, \ldots).$$

If two of the three sequences are invariant sequences, then the other sequence is also an invariant sequence.

Proof: Let $a^*(x)$, $b^*(x)$, and $c^*(x)$ be the exponential generating functions of $a(x)$, $b(x)$, and $c(x)$, respectively. It is clear that $a^*(x)b^*(x) = c^*(2x)$. So

$$c^*(2x)e^{-x} = a^*(x)e^{-x/2} \cdot b^*(x)e^{-x/2}.$$

This, together with Theorem 3.2 yields the result.

Corollary 5.3: Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying

$$\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k} = 1 \quad (n = 0, 1, 2, \ldots).$$

Then $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

Proof: Taking $c_n = 1/2^n$ in Theorem 5.2 gives the result.

Theorem 5.3: If $\{A_n\} \in IS$ with $A_0 \neq 0$ and if $\{a_n\}$ is given by

$$a_0 A_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} A_k a_{n-k} = 0 \quad (n = 1, 2, 3, \dots),$$

then $\{a_{n+2}\} \in IS$ and $\{\Sigma_{k=0}^{n} a_k\} \in IS$.

Proof: Let $a(x)$ and $A(x)$ be the generating functions of $\{a_n\}$ and $\{A_n\}$, respectively. It is clear that $a(x)A(x) = 1$. Set $a_1(x) = \Sigma_{n=0}^{\infty}(\Sigma_{k=0}^{n} a_k)x^n$. Then

$$a_1(x) = \frac{1}{1-x}a(x) = \frac{1}{(1-x)A(x)}.$$

Since $A(\frac{x}{x-1}) = (1-x)A(x)$ by Theorem 3.1, from the above we see that

$$a_1\left(\frac{x}{x-1}\right) = \frac{1}{1-x/(x-1)} \cdot \frac{1}{(1-x)A(x)} = (1-x)a_1(x).$$

Now, applying Theorem 3.1, we find $\{\Sigma_{k=0}^{n} a_k\} \in IS$ and so $\{a_{n+2}\} \in IS$ by Corollary 3.1(c).

Theorem 5.4: If $\{A_n\} \in IS$ with $A_0 \neq 0$ and if $\{a_n\}$ is given by

$$a_0 A_0 = 2 \quad \text{and} \quad \sum_{k=0}^{n} A_k a_{n-k} = 1 \quad (n = 1, 2, 3, \dots),$$

then $\{a_{n+1}\} \in IIS$ and $\{na_n\} \in IS$.

Proof: Let $a(x)$ and $A(x)$ be the generating functions of $\{a_n\}$ and $\{A_n\}$, respectively. It is obvious that $a(x)A(x) = 1 + \frac{1}{1-x}$. Since $A(\frac{x}{x-1}) = (1-x)A(x)$ by Theorem 3.1, we find

$$a\left(\frac{x}{x-1}\right) = \frac{2-x}{(1-x)A(x)} = a(x).$$

Set $a_0(x) = \Sigma_{n=0}^{\infty} a_{n+1}x^n$ and $a_1(x) = \Sigma_{n=0}^{\infty} na_n x^n$. Then $a_0(x) = (a(x) - a_0)/x$ and $a_1(x) = xa'(x)$, where $a'x$ $(= \Sigma_{n=1}^{\infty} na_n x^{n-1})$ is the formal derivative of $a(x)$. Hence, by the above, we get

$$a_0\left(\frac{x}{x-1}\right) = \left(a\left(\frac{x}{x-1}\right) - a_0\right)(x-1)/x = (x-1)a_0(x)$$

and

$$(1-x)a_1(x) = (1-x)xa'\left(\frac{x}{x-1}\right)\left(\frac{x}{x-1}\right)' = \frac{x}{x-1}a'\left(\frac{x}{x-1}\right) = a_1\left(\frac{x}{x-1}\right).$$

This implies that $\{a_{n+1}\} \in IIS$ and $\{na_n\} \in IS$ by Theorem 3.1.

Theorem 5.5: Let $\{F_n\}$ be the Fibonacci sequence. If $\{a_n\}$ and $\{b_n\}$ satisfy the relation $a_n = \Sigma_{k=0}^{n} F_{k-1} b_{n-k}$ $(n = 0, 1, 2, \dots)$, then $\{a_n\} \in IS$ if and only if $\{b_n\} \in IS$.

Proof: It is well known that $\Sigma_{n=0}^{\infty} F_n x^n = x/(1-x-x^2)$. Thus,

$$\sum_{n=0}^{\infty} F_{n+1}x^n = \frac{1}{1-x-x^2},$$

and therefore

$$\sum_{n=0}^{\infty} F_{n-1}x^n = \sum_{n=0}^{\infty} F_{n+1}x^n - \sum_{n=0}^{\infty} F_n x^n = \frac{1-x}{1-x-x^2}.$$

Let $a(x)$ and $b(x)$ denote the generating functions of $\{a_n\}$ and $\{b_n\}$, respectively. From the relation $a_n = \sum_{k=0}^{n} F_{k-1} b_{n-k}$ $(n = 0, 1, 2, \dots)$, we find

$$a(x) = \frac{1-x}{1-x-x^2} b(x).$$

Thus,

$$a\left(\frac{x}{x-1}\right) = \frac{1-\frac{x}{x-1}}{1-\frac{x}{x-1}-\left(\frac{x}{x-1}\right)^2} b\left(\frac{x}{x-1}\right) = \frac{1-x}{1-x-x^2} b\left(\frac{x}{x-1}\right).$$

Hence, by Theorem 3.1,

$$\{a_n\} \in IS \Leftrightarrow a\left(\frac{x}{x-1}\right) = (1-x)a(x) \Leftrightarrow b\left(\frac{x}{x-1}\right) = (1-x)b(x) \Leftrightarrow \{b_n\} \in IS.$$

This proves the theorem.

Remark 5.1: One can easily prove the following inversion formula.

$$a_n = \sum_{k=0}^{n} F_{k-1} b_{n-k} \quad (n = 0, 1, 2, \dots) \Leftrightarrow b_n = a_n - \sum_{k=0}^{n-2} a_k \quad (n = 0, 1, 2, \dots).$$

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In 1992, Chang-Fu Wang conjectured that, if $a_n = (-1)^n \int_0^1 \binom{x}{n} dx$ $(n = 0, 1, 2, \dots)$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+2} = a_{n+2} \quad (n = 0, 1, 2, \dots).$$

Later, Hou-Rong Qin proved the conjecture, and Zhi-Wei Sun showed that $\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0$ $(n \geq 1)$. Inspired by their work, I began to study invariant sequences, so I am grateful to them for initial enlightenment.

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