ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-925 ***Proposed by José Luis Díaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Prove that $\sum_{k=0}^{n} F_{k+1}^2$ divides

$$\sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] \text{ for } n \ge 0.$$

B-926 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

If $1 < a < \alpha$, evaluate

$$\lim_{n\to\infty}\left(a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_n}} - a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots\frac{1}{F_{n-1}}}\right).$$

B-927 ***Proposed by R. S. Melham, University of Technology, Sydney, Australia***

G. Candido ["A Relationship between the Fourth Powers of the Terms of the Fibonacci Series," *Scripta Mathematica* **17.3-4** (1951):230] gave the following fourth-power relation:

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) = (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2.$$

Generalize this relation to the sequence defined for all integers n by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a, W_1 = b,$$

B-928 ***Proposed by H.-J. Seiffert, Berlin, Germany***

The Fibonacci polynomials are defined by $F_0(x)=0$, $F_1(x)=1$, $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$ for $n\geq 0$. Show that, for all complex numbers x and all nonnegative integers n,

$$F_{2n+1}(x)=\sum_{k=0}^{n}(-1)^{\lceil k/2\rceil}\binom{n-\lceil k/2\rceil}{\lfloor k/2\rfloor}(x^2+2)^{n-k},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor- and ceiling-function, respectively.

B-929 ***Proposed by Harvey J. Hindin, Huntington Station, NY***

Prove that:

A) $F_{2N}=(1/5^{1/2})\sum_{K=0}^{2N-1}P_K(5^{1/2}/2)P_{2N-1-K}(5^{1/2}/2)$ for $N\geq 1$

and

B) $L_{2N+1}=\sum_{K=0}^{2N}P_K(5^{1/2}/2)P_{2N-K}(5^{1/2}/2)$ for $N\geq 0$,

where $P_K(x)$ is the Legendre polynomial given by $P_0(x)=1$, $P_1(x)=x$, and the recurrence relation $(K+1)P_{K+1}(x)=(2K+1)xP_K(x)-KP_{K-1}(x)$.

SOLUTIONS

Divisible or Not Divisible; That Is, by 5

B-911 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India (Vol. 39, no. 1, February 2001)***

Determine whether $L_n+2(-1)^m L_{n-2m-1}$ is divisible by 5 for all positive integers m and n.

Solution by Pantelimon Stănică, Montgomery, AL

We prove that the expression is divisible by 5 for all positive integers m, n. Formula (17b) on page 177 in S. Vajda'a *Fibonacci & Lucas Numbers, and the Golden Section* (Ellis Horwood) states: $L_{p+k}-(-1)^k L_{p-k}=5F_pF_k$. Taking $p=n-m$, $k=m+1$, and $p=n-m-1$, $k=m$, we get $L_{n+1}-(-1)^{m+1}L_{n-2m-1}=5F_{n-m}F_{m+1}$ and $L_{n-1}-(-1)^m L_{n-2m-1}=5F_{n-m-1}F_m$. Subtracting the second formula from the first, and using the definition of L_n, we obtain

$$L_n+2(-1)^m L_{n-2m-1}=5(F_{n-m}F_{m+1}-F_{n-m-1}F_m),$$

which implies the claim.

Also solved by Brian D. Beasley, Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Russell Hendel, Walther Janous, Harris Kwong, Carl Libis, H.-J. Seiffert, James Sellers, and the proposer.

From a Product to a Sum

B-912 ***Proposed by the editor (Vol. 39, no. 1, February 2001)***

Express $F_{n+(n \bmod 2)}\cdot L_{n+1-(n \bmod 2)}$ as a sum of Fibonacci numbers.

Solution by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC

The following formulas from [1] will be used:

$$(\mathrm{I}_6)\ \sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1 \quad \text{and} \quad (\mathrm{I}_{23})\ F_{n+p} - F_{n-p} = F_n L_p \ \text{ if } p \text{ is odd.}$$

Case 1: n even $\Rightarrow n \bmod 2 = 0$. So

$$F_n \cdot L_{n+1} = F_{n+n+1} - F_{n-n-1} = F_{2n+1} - F_{-1} = F_{2n+1} - 1 = \sum_{j=1}^{n} F_{2j}.$$

Case 2: n odd $\Rightarrow n \bmod 2 = 1$. So

$$F_{n+1} \cdot L_n = F_{2n+1} - F_1 = F_{2n+1} - 1 = \sum_{j=1}^{n} F_{2j}.$$

Thus,

$$F_{n+(n \bmod 2)} \cdot L_{n+1-(n \bmod 2)} = \sum_{j=1}^{n} F_{2j}.$$

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Brian D. Beasley (3 solutions), Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Pentti Haukkanen, Russell Hendel, Steve Hennagin, Walther Janous, Harris Kwong, Carl Libis, H.-J.-Seiffert, James Sellers, Pantelimon Stănică, and the proposer.

A "Constant" Search

B-913 ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA (Vol. 39, no. 1, February 2001)***

Fix an integer $k \geq 1$. The Fibonacci numbers satisfy an "accelerated" recurrence of the form $F_{n2^k} = \alpha_k F_{(n-1)2^k} - F_{(n-2)2^k}$ $(n = 2, 3, \ldots)$ with $F_0 = 0$ and F_{2^k} to start the recurrence. For example, when $k = 1$, we have $F_{2n} = 3F_{2(n-1)} - F_{2(n-2)}$ $(n = 2, 3, \ldots; F_0 = 0; F_2 = 1)$.

a. Find the constant α_k by identifying it as a certain member of a sequence that is known by readers of these pages.

b. Generalize this result by similarly identifying the constant β_m for which the accelerated recurrence $F_{mn+h} = \beta_m F_{m(n-1)+h} + (-1)^{m+1} F_{m(n-2)+h}$, with appropriate initial conditions, holds.

Solution by N. Gauthier, Kingston, ON

Case a is deduced from Case b by setting $h = 0$ and $m = 2^k$ for k a positive integer, so we solve Case b only. The sought answer is $\beta_m = L_m$ for values of n such that $m(n-1)+h \neq 0$; for $m(n-1)+h = 0$, β_m can be arbitrary but finite, since $F_0 = 0$. The former is of interest and we have, from the definition, that

$$\beta_m = \frac{[\alpha^{mn+h} + (-1)^m \alpha^{m(n-2)+h}] - [\beta^{mn+h} + (-1)^m \beta^{m(n-2)+h}]}{[\alpha^{m(n-1)+h} - \beta^{m(n-1)+h}]}$$

$$= \frac{[\alpha^{m(n-1)+h}(\alpha^m + (-1)^m \alpha^{-m}) - \beta^{m(n-1)+h}(\beta^m + (-1)^m \beta^{-m})]}{[\alpha^{m(n-1)+h} - \beta^{m(n-1)+h}]}$$
$$= [\alpha^m + \beta^m] = L_m$$

since $\alpha^{-m} = (-1)^m \beta^m$ and vice versa. This completes the proof.

Also solved by Brian D. Beasley, Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Walther Janous, Harris Kwong, H.-J. Seiffert, Pantelimon Stănică, and the proposer.

A "Product and a Sum" Inequality

B-914 *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 1, February 2001)*

Let $n \geq 2$ be an integer. Prove that

$$\prod_{k=2}^{n}\left\{\sum_{j=1}^{k}\frac{1}{(F_{k+2}-F_j-1)^2}\right\} \geq \frac{1}{F_2F_{n+1}}\left(\frac{n}{F_3F_4\ldots F_n}\right)^2.$$

Solution by H.-J. Seiffert, Berlin, Germany

We first prove that

$$kF_kF_{k+1} \geq (F_{k+2}-1)^2 \text{ for } k \geq 1. \tag{1}$$

For $k = 1, 2, 3, 4, 5, 6,$ and 7, this can be verified directly. If $k \geq 8$, then $F_{k+2} < 2F_{k+1} < 4F_k$, and therefore $kF_kF_{k+1} > (k/8)F_{k+2}^2 \geq F_{k+2}^2 > (F_{k+2}-1)^2$.

Let $k \geq 2$. The function

$$f(x) = \frac{1}{(F_{k+2}-x-1)^2},\ 0 \leq x \leq F_k,$$

is convex, as is seen from its second derivative. Applying Jensen's Inequality gives

$$\sum_{j=1}^{k} f(F_j) \geq kf\left(\frac{1}{k}\sum_{j=1}^{k}F_j\right).$$

From (I_1) of [1], we know that $\sum_{j=1}^{k}F_j = F_{k+2}-1$. Hence,

$$\sum_{j=1}^{k}\frac{1}{(F_{k+2}-F_j-1)^2} \geq \left(\frac{k}{k-1}\right)^2\frac{k}{(F_{k+2}-1)^2},$$

which, by (1), may be weakened to

$$\sum_{j=1}^{k}\frac{1}{(F_{k+2}-F_j-1)^2} \geq \left(\frac{k}{k-1}\right)^2\frac{1}{F_kF_{k+1}}.$$

Taking the product over $k = 2, 3, \ldots, n$, $n \geq 2$, we obtain the desired inequality.

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Paul Bruckman, Walther Janous, and the proposer.

Editor's Comment: *Walther Janous actually improved the inequality by elementary means and showed the right-hand side to be greater than* $n!/2^{2(n-1)}\prod_{k=2}^{n}(F_{k+1}-1)^2$.

A "Double Sum" Inequality

B-915 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 39, no. 1, February 2001)***

If $|x|\le 1$, prove that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_jx^{i-1}\right| < n^3.$$

Solution by Paul Bruckman, Sacramento, CA

A stronger result is actually true, namely:

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_jx^{i-1}\right| \le n(n+1)/2, \text{ whenever } |x|\le 1.$$

Let $G(x,n)$ denote the quantity within the absolute value bars. Then

$$|G(x,n)|\le|G(1,n)|=\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_j\le\sum_{i=1}^{n} i\sum_{j=1}^{\infty}2^{-j-1}F_j.$$

Now $\sum_{j=1}^{\infty}u^{j-1}F_j=(1-u-u^2)^{-1}$, provided $|u|\le\alpha^{-1}$. Setting $u=1/2$, we obtain $\sum_{j=1}^{\infty}2^{-j-1}F_j=1$. Thus,

$$|G(x,n)|\le\sum_{i=1}^{n} i=n(n+1)/2.$$

Note that $n(n+1)/2\le n^3$, with equality iff $n=1$. Since $G(x,1)=1/4$, we see that the result indicated in the statement of the problem is certainly true.

Also solved by Ovidiu Furdui, H.-J. Seiffert, and the proposer.

A Response to Gauthier's Comments on the Bruckman Conjecture

A Comment by Paul Bruckman

First, I would like to make a slight correction. Although I appreciate being referred to as "Professor Bruckman," I must regretfully inform the world that I am no longer teaching math, having returned to consulting work for a private firm.

Secondly, I am sincerely flattered to have my name associated with a certain set of polynomials (the $P_r(n)$ of Dr. Gauthier's note). Before accepting this honor, however, I would like to be sure that these polynomials are indeed new in the literature; I would be loath to usurp someone else's rightful niche in mathematical history.

I am grateful to Dr. Gauthier for pointing out the corrected version of my conjecture. I might have discovered this for myself, had I taken the time and effort to develop the correct polynomial expressions, as Dr. Gauthier has obviously done.

It should be mentioned that there is an advanced problem proposed by Dr. Gauthier (H-568) in the last issue [*The Fibonacci Quarterly* **38.5** (2000):473; corrected **39.1** (2001):91-92] which is highly interesting and bears some superficial resemblance to my problem B-871 [**37.1** (1999): 85]. However, unlike the polynomials $P_r(n)$, the "Gauthier functions" $f_m(n)$ are rational functions. I have submitted my solution for Problem H-568 to the Advanced Problems Editor.

The remainder of this letter is devoted to indicating some of my subsequent research in response to Dr. Gauthier's comments.

Following Gauthier's notation, we may define the functions $P_r(n)$ as follows:

$$P_r(n) \equiv \binom{2n}{n}^{-1} \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^{2r-1}. \tag{1}$$

As Dr. Gauthier correctly pointed out, what I *should* have originally conjectures was:

$$P_r(n) \text{ is a polynomial in } n \text{ of degree } r \tag{2}$$

(that is, leaving out the erroneous modifier "monic").

Actually, we can prove a somewhat stronger result than (2), namely: $P_r(n)$ is a polynomial in n of degree r, with its first two leading terms given by

$$P_r(n) = (r-1)!\,n^r - (r-2)!\binom{r}{3} n^{r-1} + \cdots. \tag{3}$$

Towards this end, we first demonstrate that the $P_r(n)$'s satisfy the recurrence relation:

$$P_{r+1}(n) = n^2(P_r(n) - P_r(n-1)),\ r = 1, 2, \ldots. \tag{4}$$

By means of (4), with $P_1(n) = n$, we readily obtain the expressions for $P_r(n)$ indicated by Gauthier in his note, for $r = 1, 2, 3, 4, 5$. The proof of (4) is straightforward, following from the definition of the $P_r(n)$. In turn, (4) implies (3), as can be demonstrated by induction.

What appears to be a more difficult problem is to obtain a general expansion (for all the terms) of $P_r(n)$. Once obtained, this might reveal other properties of the $P_r(n)$, and may possibly demonstrate that they are special cases of well-studied polynomials with known properties.

If it should turn out that these are indeed new polynomials, they may be expected to yield additional research and should generate further interest in their properties. It already seems interesting enough that the special form given in the definitions of the polynomials $P_r(n)$ and $f_m(n)$ (as given in Dr. Gauthier's note and in H-568, respectively) should yield polynomial functions and rational functions, respectively.

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