EXTENSIONS OF RECURRENCE RELATIONS

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The purpose of this article is to investigate analytic extensions of F_n and L_n to the complex plane. We shall begin by considering a particular extension. Later we will consider alternate extensions. We begin with the following notation

$$\alpha = (1+\sqrt{5})/2 \quad \text{and} \quad \beta = (1-\sqrt{5})/2 \quad .$$

Since $\beta < 0$ we adopt the convention $\beta = e^{i\pi}(-\beta)$.

With these conventions, we shall make the following definitions:

The Fibonacci Function, $F(z) = 1/\sqrt{5}\,(\alpha^z - \beta^z)$

The Lucas Function, $L(z) = \alpha^z + \beta^z$.

Note that $F(n) = F_n$ and $L(n) = L_n$, where n denotes an integer.

I Periodic Properties of $F(z)$ and $L(z)$

Theorem 1. α^z is periodic with period $2\pi i/\ln\alpha = p_\alpha$.

Proof. $\alpha^{z+p_\alpha} = \alpha^z e^{2\pi i} = \alpha^z$.

Theorem 2. β^z is periodic with period $2\pi/(\ln^2\alpha + \pi^2)(\pi - i\ln\alpha) = p_\beta$.

Proof. Since $-\ln\alpha = \ln(-\beta)$, we have

$$\beta^{z+p_\beta} = \beta^z e^{2\pi i} = \beta^z \quad .$$

Theorem 3. $F(z)$ and $L(z)$ are not periodic.

Proof. Deny! Assume $F(z)$ has period ω. $F(0) = 0 = F(\omega)$ implies $\alpha^\omega = \beta^\omega$.

Thus $F(z+\omega) = 1/\sqrt{5}\,\alpha^\omega(\alpha^z - \beta^z)$.

Hence $\alpha^\omega = 1$, so $\mathrm{Re}(\omega) = 0$. Then $\beta^\omega \neq 1$ unless $\omega = 0$.

The proof for $L(z)$ is similar.

II Zeroes of $F(z)$ and $L(z)$.

Theorem 4. The zeroes of $F(z)$ are

$$4k\pi\ln\alpha/(4\ln^2\alpha + \pi^2)(-\pi/2\ln\alpha + i) \quad .$$

Proof. Note that this theorem implies the only real zero of $F(z)$ is 0.

$F(z) = 0$ implies $(\alpha/\beta)^z = 1 = e^{2k\pi i}$, k an integer.

Setting $z = x + iy$ and collecting real and imaginary parts and equating, the result follows.

The moduli of the zeroes are $|2k|\pi/\sqrt{4\ln^2\alpha + \pi^2}$.

Theorem 5. The zeroes of $L(z)$ are

$2(2k+1)\ln\alpha/(4\ln^2\alpha + \pi^2)(-\pi/2\ln\alpha + i) = z_k$,

where k is an integer.

Proof. Note that this theorem implies $L(z)$ has no real zeroes. Write $-1 = e^{(2k+1)\pi i}$ and proceed as above.

The moduli of the zeroes are $|2k+1|\pi/\sqrt{4\ln^2\alpha + \pi^2}$.

Observe that all of the zeroes of $L(z)$ and $F(z)$ are on the ray $\theta = \mathrm{Arctan}\,((-2\ln\alpha)/\pi) \sim -20^\circ$.

III Behavior of $F(z)$ and $L(z)$ on the real axis.

Theorem 6. On the real axis, the only real values of $F(z)$ and $L(z)$ are at $z = n$ (an integer), that is, F_n and L_n.

Proof. Since $y = 0$, $\alpha^z = \alpha^x$, $\beta^z = e^{-x\ln\alpha + \pi x i}$;

$\mathrm{Im}\,F(z) = \mathrm{Im}\,L(z) = 0$ yields

$-1/\sqrt{5}\,e^{-x\ln\alpha}\sin\pi x = 0$ or

$e^{-x\ln\alpha}\sin\pi x = 0$.

Hence $x = k$, k an integer.

(It is not too difficult to show that the only lattice points with real images for $F(z)$ are on the real axis.)

IV Identities Satisfied by $F(z)$ and $L(z)$.

Many of the identities of F_n and L_n carry over to $F(z)$ and $L(z)$. We shall list a few of them. They are easy to verify.

a. $F(z+2) = F(z+1) + F(z)$

b. $L(z+2) = L(z+1) + L(z)$

c. $F(z+1)F(z-1) - F^2(z) = e^{\pi iz}$

d. $L^2(z) - 5F^2(z) = 4e^{\pi iz}$

e. $F(-z) = -e^{\pi iz} F(z)$

f. $F(z)L(z) = F(2z)$

g. $F(z+w) = F(z-1)F(w) + F(z)F(w+1)$

h. $F(3z) = F^3(z+1) + F^3(z) - F^3(z-1)$

i. $\lim_{x\to\infty} F(x)/F(x+1) =$

$\lim_{y\to\infty} F(iy)/F(iy+1) = -\beta$

In general, $(-1)^n$ in an identity for F_n and L_n carries over to $e^{\pi iz}$. The identities which do not carry over to $F(z)$ and $L(z)$ are those which only make sense for integral argument. That is, those which involve binomial coefficients, etc.

V Analytic Properties of $F(z)$ and $L(z)$.

Note that our convention for β implies $\ln\beta = \pi i + \ln(-\beta)$. It is thus immediate that $F(z)$ and $L(z)$ are holomorphic in the plane (entire functions).

From the Taylor formula, we have for any finite z,

$$F(z) = 1/\sqrt{5} \sum_{k=0}^{\infty} \left\{\left[(\ln^k\alpha)\alpha^w - (\ln^k\beta)\beta^w\right]/k!\right\} (z-w)^k \quad \text{and}$$

$$L(z) = \sum_{k=0}^{\infty} \left\{\left[(\ln^k\alpha)\alpha^w + (\ln^k\beta)\beta^w\right]/k!\right\} (z-w)^k .$$

Note the results when these are used with $w = 0$ and $z = n$ or with $w = n-1$ and $z = n$.

$$F_n = 1/\sqrt{5} \sum_{k=0}^{\infty} \left[(\ln^k\alpha)\alpha^{n-1} - (\ln^k\beta)\beta^{n-1}\right]/k! .$$

This is, I believe, a new representation for F_n. The Hadamard Factorization theorem can be used to express $L(z)$ as a canonical product. As in theorem 5, let z_k represent a zero of $L(z)$. Renumber z_k as follows:

$$k = -1,\ 0,\ -2,\ 1,\ -3,\ 2,\ \ldots$$
$$n = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

Now $|z_n| \le |z_{n+1}|$ and $|z_n| = 0(n)$. It is easy to see that $L(z)$ is of order and genus 1 and we have

$$L(z) = e^{cz} \prod_{n=1}^{\infty} (1 - z/z_n)\, e^{z/z_n}\,, \quad \text{where}$$

$$c = - \sum_{n=1}^{\infty} \left[\ln(1 - 1/z_n) + 1/z_n\right] \,.$$

We shall now discuss exceptional values of $F(z)$ and $L(z)$. Since $F(z)$ and $L(z)$ are entire functions with essential singularities at ∞, by Picard's theorem, they must take on every value, except possibly one, and infinite number of times.

$$\lim_{x\to\infty} L(x-ix) = \lim_{x\to\infty} F(x-ix) = 0$$

Thus 0 is an asymptotic value for $F(z)$ and $L(z)$.

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} F(x) = \infty \text{ and } \infty$$

is an asymptotic value for $F(z)$ and $L(z)$.

Ahlfors has shown that entire functions of order P have at most $2P$ asymptotic values [1]. Further, if an integral function has z as an exceptional value, then z is an asymptotic value [2]. Now 0 is not an exceptional value for $F(z)$ or $L(z)$; Part II. Hence $F(z)$ and $L(z)$ have no finite exceptional values.

Thus the Fibonacci Prime Conjecture is trivial in the complex plane; that is, there are an infinite number of Fibonacci images which are distinct primes. It is conceivable that a knowledge of the distribution of prime images might yield a resolution of this conjecture, although this problem is probably more difficult than the conjecture itself. Poisson's formulae for real and imaginary parts of $F(z)$ might be useful, but the integrals are horrible Fresnel type integrals [3].

A characterization of the point set corresponding to $\operatorname{Im} F(z) = 0$ should present an interesting problem. Graphs of $\{z \mid \operatorname{Re} F(z) = 0\}$,

$\{z \mid \operatorname{Im} F(z) = 0\}$, $\{z \mid |F(z)| = M\}$ in some neighborhood of the origin should yield interesting diagrams.

VI Alternate Extensions.

There are an infinite number of extensions of F_n and L_n to entire functions in the complex plane. If the functional equation

$$G(z+2) = G(z+1) + G(z); \quad G(0) = 0, \ G(1) = 1 \quad ,$$

is used as a starting point, it appears that very little can be established. However it is possible to obtain extensions which are real at every point of the real axis. Consider, for example,

$$F_1(z) = 1/\sqrt{5}\left[\alpha^z - \sin\left(\frac{2z+1}{2}\pi\right)(-\beta)^z\right] \quad .$$

Note that $F_1(z)$ satisfies the relation,

$$F_1(z+1)F_1(z-1) - F_1^2(z) = \sin(2z+1)\pi/2 \quad .$$

$F_1(z)$ is an entire function and has zeroes on the negative real axis and $F_1(n) = F_n$, n an integer.

Another type of extension is,

$$F_2(z) = e^{2\pi i z} F(z) + \sin \pi z \quad .$$

Practically none of the above theorems hold for arbitrary extensions. The following construction seems to indicate that F_n could be extended to a periodic entire function in the complex plane. Consider the rectangle, R, in the complex plane bounded by

$$(1,0), \ (1,1), \ (-1,1), \ (-1,0) \quad .$$

Select a function, $F_3(z)$, subject to the following conditions:

a. $F_3(0) = 0$

b. $F_3(-1+iy) + F_3(iy) = F_3(1+iy); \quad y \in [0,1]$

c. $F_3(x) = F_3(x+i); \quad x \in [-1,1]$

d. $F_3(-1) = F_3(1) = 1$

e. $F_3(z)$, analytic on R .

Extend $F_3(z)$ vertically by periodicity and horizontally by the functional equation, $F_3(z+2) = F_3(z+1) + F_3(z)$. The extension would be an entire function with period i and $F_3(n) = F_n$, n an integer.

REMARKS

Selection of a proper extension for $F(n)$ should, via the machinery of Analytic Function theory, put a powerful wrench on the Fibonacci Prime Conjecture.

REFERENCES

1. E. Titchmarsh, The Theory of Functions, 2nd ed. (1952), p. 284b.
2. Ibid, p. 284a.
3. Ibid, pp. 124-125.

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CORRECTIONS

"Binomial Coefficients, the Bracket Function, and Compositions with Relatively Prime Summands" by H. W. Gould, Fibonacci Quarterly, 2(1964), pp. 241-260.

Page 241. The second paragraph should begin: "Indeed this result is equivalent to the identical congruence $(1 - x)^p \equiv 1 - x^p \pmod p$ ..."

Page 245. In Theorem 3 it is necessary to require $a_i > 0$.

Page 257. Line after relation (48), replace "out" by "our".

Page 251. Line 9 from bottom, for "as" read "an".

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