# SOME DETERMINANTS CONTAINING POWERS OF FIBONACCI NUMBERS

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1. F. D. Parker (Problem H-46, this Quarterly, Vol. 2 (1964), p. 303) has proposed the evaluation of the determinant

$$|F^4_{n+i+j-2}| \quad (i, j = 1, 2, 3, 4, 5,) .$$

This suggests the more general problem of evaluating

$$D_k = D_{k,n} = |F^k_{n+r+s}| \quad (r, s = 0, 1, \ldots, k) .$$

We shall show that

$$(1) \qquad D_k = (-1)^{\frac{1}{2}k(k+1)(n+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \ldots F_k)^2 .$$

For example

$$D_1 = (-1)^{n+1}, \quad D_2 = (-1)^{n+1} 2, \quad D_3 = 36 ,$$

$$D_4 = 13824.$$

To prove (1) we consider the quadratic form

$$Q = \sum_{r,s=0}^{k} F^k_{n+r+s} u_r u_s .$$

Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}) ,$$

we have

$$(\alpha-\beta)^k Q = \sum_{r,s=0}^{k} u_r u_s \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \alpha^{(n+r+s)j} \beta^{(n-r-s)(k-j)}$$

$$=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\alpha^{nj}\beta^{n(k-j)}\sum_{r,\,s=0}^{k}\alpha^{(r+s)j}\beta^{(r+s)(k-j)}u_r u_s$$

$$=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\alpha^{nj}\beta^{n(k-j)}\left(\sum_{r=0}^{k}\alpha^{rj}\beta^{r(k-j)}u_r\right)^2 .$$

If we put

$$v_j=\sum_{r=0}^{k}(\alpha^j\beta^{k-j})^r u_r , \tag{2}$$

it is clear that

$$(\alpha-\beta)^k Q=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\alpha^{nj}\beta^{n(k-j)}v_j^2 . \tag{3}$$

Thus by means of the linear transformation (2), we have reduced $Q$ to diagonal form. If $\Delta$ denotes the determinant of the linear transformation (2), it follows from (3) that

$$D_k=(-1)^{\frac{1}{2}k(k+1)}\prod_{j=0}^{k}\binom{k}{j}\cdot(\alpha-\beta)^{-k(k+1)}(\alpha\beta)^{\frac{1}{2}nk(k+1)}\Delta^2 . \tag{4}$$

Now

$$\Delta=\left|(\alpha^r\beta^{k-r})^s\right| \quad (r,\ s=0,1,\ldots,k) .$$

Since this is a Vandermonde determinant we get

$$\Delta=\prod_{0\le r<s\le k}(\alpha^s\beta^{k-s}-\alpha^r\beta^{k-r})$$

$$=\prod_{0\le r<s\le k}\alpha^r\beta^{k-s}(\alpha-\beta)F_{s-r}$$

$$=(\alpha-\beta)^{\frac{1}{2}k(k+1)}\prod_{r=0}^{k-1}\prod_{s=0}^{k-r}\alpha^r\beta^{k-r-s}F_s$$

$$= (\alpha-\beta)^{\frac{1}{2}k(k+1)} \prod_{r=0}^{k-1} \alpha^{r(k-r)} \beta^{\frac{1}{2}(k-r)(k-r-1)} \prod_{s=1}^{k-r} F_s$$

$$= (\alpha-\beta)^{\frac{1}{2}k(k+1)} (\alpha-\beta)^{\frac{1}{6}k(k+1)(k-1)} F_1^k F_2^{k-1} \cdots F_k .$$

Therefore (4) becomes

$$D_k = (-1)^{\frac{1}{2}k(k+1)(n+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \cdots F_k)^2 .$$

This completes the proof of (1).

2. As for the determinant

$$D_k(L) = \left| L^k_{n+r+s} \right| \quad (r, s = 0, 1, \ldots, k) \quad ,$$

consideration of the quadratic form

$$\sum_{r,s=0}^{k} L^k_{n+r+s} u_r u_s$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \sum_{r,s=0}^{k} \alpha^{(r+s)j} \beta^{(r+s)(k-j)} u_r u_s$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} \left( \sum_{r=0}^{k} \alpha^{rj} \beta^{r(k-j)} u_r \right)^2$$

yields

$$D_k(L) = \prod_{j=0}^{k} \binom{k}{j} \alpha^{nj} \beta^{n(k-j)} . \Delta^2$$

$$= (-1)^{\frac{1}{2}nk(k+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (\alpha-\beta)^{k(k+1)} (F_1^k F_2^{k-1} \cdots F_k)^2 .$$

It follows that

$$(5) \qquad D_k(L) = (-1)^{\frac{1}{2}nk(k+1)} 5^{\frac{1}{2}k(k+1)} \prod_{j=0}^{k} \binom{k}{j} \cdot (F_1^k F_2^{k-1} \cdots F_k)^2 .$$

3. Formulas (1) and (5) can be generalized in an obvious way. Consider the sequence $\{W_n\}$ defined by

$$W_{n+1} = p\, W_n - q\, W_{n-1} \qquad (n \geq 1) ,$$

where $W_0$, $W_1$ are assigned. Put

$$D_k(W) = |W_{n+r+s}^k| \qquad (r,\ s=0,\ 1,\ \ldots,\ k) .$$

If $p^2 - 4q \neq 0$ and

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} , \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

then $W_n = A\,\alpha^n + B\beta^n$, where

$$A = \frac{W_1 - \beta W_0}{\alpha - \beta} , \quad B = \frac{\alpha W_0 - W_1}{\alpha - \beta} , \quad AB = - \frac{W_1^2 - pW_0W_1 + qW_0^2}{p^2 - 4q} .$$

We find that

$$(6) \qquad D_k(W) = (-1)^{\frac{1}{2}k(k+1)} q^{\frac{1}{2}nk(k+1) + \frac{1}{3}k(k+1)(k-1)} \prod_{j=0}^{k} \binom{k}{j}$$

$$\cdot\ (W_1^2 - pW_0W_1 + qW_0^2)^{\frac{1}{2}k(k+1)} (U_1^k U_2^{k-1} \cdots U_k)^2 ,$$

where

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

Indeed (6) holds also when $p^2 - 4q = 0$, provided we now take $U_n = n(p/2)^{n-1}$. This can be proved directly in the following way. We have

$$W_n = \left(\frac{p}{2}\right)^n (A + Bn) ,$$

where $A = W_0$, $pB = 2W_1 - pW_0$. Then

$$D_k(W) = \left|\left(\frac{p}{2}\right)^{(n+r+s)k} (A + B(n+r+s))^k\right| \quad (r,\ s=0,\ 1,\ \ldots,\ k)$$

$$= \left(\frac{p}{2}\right)^{nk(k+1)+k^2(k+1)} \left|(A + B(n+r+s))^k\right| \quad .$$

We recall that for a determinant of the type

$$\left|u_{n+r+s}\right| \qquad (r,\ s=0,\ 1,\ \ldots,\ k)$$

we have

$$\left|u_{n+r+s}\right| = \left|\Delta^{r+s} u_n\right| \qquad (r,\ s=0,\ 1,\ \ldots,\ k) \quad ,$$

where $\Delta$ is the usual finite difference operator. (See for example [1, p. 103].) In the present instance $u_n = (A+Bn)^k$, so that

$$\Delta^{r+s} u_n = 0 \qquad (r + s > k), \qquad \Delta^k u_n = k!\ B^k \quad .$$

It follows that

$$\left|(A+B(n+r+s))^k\right| = (-1)^{\frac{1}{2}k(k+1)} (k!)^{k+1}\ B^{k(k+1)}$$

and therefore

$$(7) \quad D_k(W) = (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+k(k+1)(k-1)} (k!)^{k+1} \left(W_1 - \frac{p}{2}W_0\right)^{k(k+1)} \quad .$$

On the other hand (6) becomes

$$D_k(W) = (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+\frac{2}{3}k(k+1)(k-1)} \prod_{j=0}^{k} \binom{k}{j}$$

$$\cdot \left(W_1 - \frac{p}{2}W_0\right)^{k(k+1)} \prod_{j=1}^{k} j^{2(k-j+1)} \left(\frac{p}{2}\right)^{2(j-1)(k-j+1)}$$

$$= (-1)^{\frac{1}{2}k(k+1)} \left(\frac{p}{2}\right)^{nk(k+1)+k(k+1)(k-1)} \left(W_1 - \frac{p}{2}W_0\right)^{k(k+1)}$$

$$\cdot \prod_{j=1}^{k} \binom{k}{j} j^{2(k-j+1)}$$

Since

$$\prod_{j=1}^{k} \binom{k}{j} j^{2(k-j+1)} = \prod_{j=1}^{k} \binom{k}{j} \left(\frac{j!}{(j-1)!}\right)^{2(k-j+1)} = \prod_{j=0}^{k} \binom{k}{j}(j!)^2$$

$$= \prod_{j=0}^{k} \frac{k!\ j!}{(k-j)!} = (k!)^{k+1} ,$$

it is clear that (6) and (7) are in agreement.

## REFERENCE

1. G. Kowalewski, Determinanten theorie, Chelsea, New York, 1948.

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