A LOWER BOUND FOR MAXIMUM ZERO-ONE DETERMINANTS

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What is the largest possible determinant of order n, if zero and one are the only entries allowed? This question, last posed by Harary [1], seems difficult.

Williamson [2] obtained the values 1, 1, 2, 3, 5, 9 for n = 1, 2, 3, 4, 5, 6; and proved the general problem equivalent to a similar Hadamard question with allowed entries 1 and -1.

Cohn [3] derived an asymptotic lower bound,

$$(n+1)^{(\frac{1}{2}-\epsilon)(n+1)}/2^n \quad ,$$

where ϵ is any positive number. The upper bound

$$(n+1)^{(n+1)/2}/2^n$$

follows from Hadamard's inequality [4], applied to the 1, -1 version of the problem.

Clements and Lindström [5] have announced the lower bound

$$(n+1)^K/2^n \quad ,$$

where $K = (n+1)(1-(\log 4/3)/\log(n+1))/2$, and the logarithms are base two.

In this note, I show that the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... is a lower bound for the sequences of maximum zero-one determinants. Also, I compare this bound with the Clements-Lindström bound.

Theorem: The maximum zero-one determinant of order n is at least as large as the n^{th} Fibonacci number.

This is proved by exhibiting zero-one matrices whose determinants are the Fibonacci numbers.

Let $a(n)$ be the row vector with n entries which are alternately one and zero, starting with one. Consider the n^{th} order matrix

$$F(n) = \begin{pmatrix} a(n) & & & & & \\ 1 & a(n-1) & & & & \\ 0 & 1 & \cdot & & & \\ \cdot & 0 & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & a(2) & \\ 0 & 0 & & & 1 & a(1) \end{pmatrix}$$

For example,

$$F(1) = (1),\ F(2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ \text{and}\ F(3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\det F(1) = 1$, $\det F(2) = 1$, and $\det F(3) = 2$.

Expanding $\det F(n)$ by the first column gives $\det F(n) =$

$$\det\begin{pmatrix} a(n-1) & & & \\ 1 & \cdot & & \\ 0 & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & a(2) & \\ 0 & & & 1 & a(1) \end{pmatrix} - \det\begin{pmatrix} 0 & a(n-2) & & & \\ 1 & a(n-2) & & & \\ 0 & 1 & a(n-3) & & \\ \cdot & 0 & 1 & \cdot & \\ \cdot & \vdots & \cdot & a(2) & \\ 0 & 0 & 0 & 1 & a(1) \end{pmatrix}$$

$$= \det F(n-1) - (-1)\det\begin{pmatrix} a(n-2) & & & & & \\ 1 & a(n-3) & & & & \\ 0 & 1 & \cdot & & & \\ \cdot & 0 & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & a(2) & \\ 0 & 0 & & & 1 & a(1) \end{pmatrix}$$

$= \det F(n-1) + \det F(n-2)$. Therefore, the sequence $\det F(n)$ is the Fibonacci sequence.

To compare this bound with the Clements-Lindström bound, examine the following table.

n	1	2	3	4	5	6	7	8	9
det F(n)	1	1	2	3	5	8	13	21	34
$(n+1)^K/2^n$	.8	.9	1.1	1.7	2.8	5.2	10.1	21.1	46.3

10	11	12	13	14	15
55	89	144	233	377	610
107.2	259.5	654.9	1,717.7	4,669	13,122

If n is greater than 8, the Clements-Lindström bound is better.

For special n, still better bounds can be found. One of Cohn's inequalities [3] becomes, for zero-one determinants,

$$M(mn-1) \geq 2^{(m-1)(n-1)} [M(m-1)]^n [M(n-1)]^m ,$$

where M(i) is the maximum determinant of order i. If mn-1 = 14 and mn-1 = 15, then

$$M(14) \geq 6,912, \quad \text{and}$$
$$M(15) \geq 131,072 .$$

The numbers in the table above were bought from Diane K. Middents for 2 palindromes.

REFERENCES

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