AN ALMOST LINEAR RECURRENCE

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A general linear recurrence with constant coefficients has the form

$$u_0 = a_1,\ u_1 = a_2, \ldots, u_{r-1} = a_r \ ;$$

$$u_n = b_1 u_{n-1} + b_2 u_{n-2} + \ldots + b_r u_{n-r}, \quad n \geq r \ .$$

The Fibonacci sequence is the simplest non-trivial case. Consider, however, the following sequence:

(1) $$\phi_0 = 1 \ ;$$

$$\phi_n = \phi_{n-1} + \phi_{[n/2]}, \quad n > 0 \ .$$

In this case, successive terms are formed from the previous one by adding the term "halfway back" in the sequence. This recurrence, which may be considered as a new kind of generalization of the Fibonacci sequence, has a number of interesting properties which we will examine here.

The sequence begins 1, 2, 4, 6, 10, 14, 20, 26, 36, ... It is easy to see that all terms except the first are even, and furthermore ϕ_n is divisible by 4 if and only if $n \equiv 2^{2k-1} \pmod{2^{2k}}$ for some $k \geq 1$. We leave it to the reader to discover further arithmetic properties of the sequence.

The sequence ϕ_n has an interesting combinatorial interpretation: ϕ_n is precisely the number of partitions of the number $2n$ into powers of 2. For example, 6 = 4 + 2 = 4 + 1 + 1 = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1, and $\phi_3 = 6$. To verify this interpretation, let P(m) be the number of partitions of m

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into powers of 2. If $2n = a_1 + a_2 + \ldots + a_k$, where $a_1 \geq a_2 \geq \cdots \geq a_k$ and each a_i is a power of 2, there are two cases: (i) $a_k = 1$; then $a_1 + \ldots + a_{k-1}$ is a partition of $2n-1$; (ii) $a_k > 1$; then $a_1/2 + a_2/2 + \ldots + a_k/2$ is a partition of n. Conversely, all partitions of 2n are obtained from partitions of $2n-1$ and n in this way, so $P(2n) = P(2n-1) + P(n)$. We also find $P(2n+1) = P(2n)$ by a similar argument; here only case (i) can arise since $2n + 1$ is an odd number. These recurrence relations for P, together with $P(1) = 1$ and $P(2) = 2$, establish the fact that $\phi_n = P(2n)$.

The same sequence also arises in other ways; the author first noticed it in connection with the solution of the recurrence relation

(1a)
$$M(0) = 0$$
$$M(n) = n + \min_{0 \leq k < n} (2M(k)+M(n-1-k))$$

for which it can be shown that $M(n) - M(n-1) = m$ if $\phi_m \leq 2n < \phi_{m+1}$, and

$$M(\tfrac{1}{2}\phi_n - 1) = \frac{n-1}{2}\phi_n - \left[\tfrac{1}{4}\phi_{2n-1}\right] .$$

Recurrences such as (1a) occur in the study of dynamic programming problems, and they will be the subject of another paper.

Let us begin our analysis of ϕ_n by noticing some of its most elementary properties. By applying the rule (1) repeatedly, we find

(2)
$$\phi_{2n+1} = 2(\phi_0 + \ldots + \phi_n) .$$

Another immediate consequence of (1) is

(3)
$$\phi_{2n}^2 - \phi_{2n+1}\phi_{2n-1} = \phi_n^2 .$$

The sequence ϕ_n grows fairly rapidly; for example,

$$\phi_{500} = 1981471878$$
$$\phi_{10000} = 2.14 \times 10^{20} .$$

In fact, we now show that ϕ_n grows more rapidly than any power of n:

Theorem 1. For any power k, there is an integer N_k such that $\phi_n > n^k$ for all $n \geq N_k$.

Proof: Let N be such that $(2^{k+1}+1) \geq (2 + \frac{1}{N})^{k+1}$, and let

$$\alpha = \min_{N \leq n \leq 2N} (\phi_n/n^{k+1}) .$$

Then by induction $\phi_n \geq \alpha n^{k+1}$ for all $n \geq N$, since this is true for $N \leq n \leq 2N$, and if $n > 2N$

$$\phi_n = \phi_{n-1} + \phi_{[n/2]} \geq \alpha(n-1)^{k+1} + [n/2]^{k+1})$$

$$\geq \alpha((n-1)^{k+1} + (\frac{n-1}{2})^{k+1}) = \alpha(1 + \frac{1}{2^{k+1}})(n-1)^{k+1} \geq \alpha(1 + \frac{1}{2N})^{k+1}(n-1)^{k+1}$$

$$\geq \alpha(1 + \frac{1}{n-1})^{k+1}(n-1)^{k+1} = \alpha n^{k+1} .$$

If we choose $N_k \geq 1/\alpha$ and $N_k \geq N$, the proof is complete.

We now consider the generating function for ϕ_n. Let

$$F(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3 + \dots \quad (4)$$

Notice that

$$(1+x)(F(x^2) = \phi_0 + \phi_0 x + \phi_1 x^2 + \phi_1 x^3 + \phi_2 x^4 + \phi_2 x^5 + \dots$$

$$= \phi_0 + (\phi_1 - \phi_0)x + (\phi_2 - \phi_1)x^2 + (\phi_3 - \phi_2)x^3 + (\phi_4 - \phi_3)x^4 + \dots$$

$$= (1-x)F(x) \quad ;$$

thus

$$F(x) = \frac{1+x}{1-x} F(x^2) = \frac{(1+x)(1+x^2)}{(1-x)(1-x^2)} F(x^4) = \dots .$$

We have therefore

$$F(x) = \frac{(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots}{(1-x)(1-x^2)(1-x^4)(1-x^8)\cdots} = \frac{1}{(1-x)^2(1-x^2)(1-x^4)(1-x^8)\cdots} \quad (5)$$

From this form of the generating function, we see that $F(x)$ converges for $|x| < 1$. (As a function of the complex variable z, $F(z)$ has the unit circle as a natural boundary.) It follows that

$$\limsup \sqrt[n]{\phi_n} = 1 \quad ,$$

i.e. the sequence ϕ_n grows more slowly than α^n for any constant $\alpha > 1$. This is in marked contrast to linear recurrences such as the Fibonacci numbers.

In the remainder of this paper we will determine the true rate of growth of the sequence ϕ_n; it will be proved by elementary methods that

$$\ln \phi_n \sim \frac{1}{\ln 4} (\ln n)^2 \quad ,$$

i.e.

$$\phi_n = e^{\frac{1}{\ln 4}(\ln n)^2 + o((\ln n)^2)} \quad . \tag{6}$$

The techniques are similar to others which have been used for determining the order of magnitude of the partition function (see [2]).

We start by observing that

$$\ln F(x) = -\ln(1-x) + \sum_{k=0}^{\infty} (-\ln(1-x^{2^k}))$$

$$= \sum_{r=1}^{\infty} \frac{x^r}{r} + \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} \frac{x^{2^k r}}{r}$$

and hence by differentiation

$$\frac{F'(x)}{F(x)} = \sum_{r=1}^{\infty} x^{r-1} + \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} 2^k x^{2^k r}$$

$$= 2 + 4x + 2x^2 + 8x^3 + 2x^4 + 4x^5 + \ldots + \theta_k x^{k-1} + \ldots$$

where θ_k is twice the highest power of 2 dividing k. Therefore

$$\frac{F'(x)}{F(x)} = (1-x)(2+6x+8x^2+16x^3+18x^4+22x^5+\ldots+\psi_k x^{k-1}+\ldots)$$

where if

$$k = 2^{a_1}+\ldots+2^{a_r},\ a_1 > a_2 > \ldots > a_r \geq 0,$$

the coefficient of x^{k-1} in the power series on the righthand side is

$$\psi_k = \theta_1 + \theta_2 + \ldots + \theta_k = a_1 2^{a_1} + \ldots + a_r 2^{a_r} + 2k \ .$$

(The reader will find the verification of this latter formula an interesting exercise in the use of the binary system.) We can estimate the magnitude of ψ_k as follows:

$$\psi_k \geq a_1 k + 2k - (2^{a_1-1} + 2\cdot 2^{a_1-2} + \ldots + a_1)$$

$$= (a_1+2)k - 2^{a_1+1} + a_1 + 2 \geq (1+\log_2 k)k - 2k \quad ;$$

hence

$$(7) \qquad k \log_2 k - k \leq \psi_k \leq k \log_2 k + 2k \quad .$$

This estimate and the monotonicity of Φ_n are the only facts about F(x) which are used in the derivation below.

Let
$$G(x) = e^{\frac{1}{\ln 4}(\ln(1-x))^2} \ .$$

Then

$$\frac{G'(x)}{G(x)} = \frac{-\log(1-x)}{\ln 2\ (1-x)} = (1-x)\left(\frac{1}{\ln 2}x + \frac{5}{2 \ln 2}x^2 + \frac{13}{3 \ln 2}x^3 + \frac{77}{12 \ln 2}x^4 + \ldots\right).$$

Since the derivative of $-\log(1-x)/(1-x)$ is $(1-\log(1-x))/(1-x)^2$, we find that the coefficient of x^{k-1} in the power series on the right is

$$(8) \qquad \chi_k = \frac{k}{\ln 2}(h_k - 1) \quad ,$$

where

(9) $$h_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k} \quad .$$

Since $h_k = \ln k + 0(1)$, we have therefore established the equations

(10) $$\frac{F'(x)}{F(x)} = (1-x)\sum_{k=1}^{\infty} \psi_k x^{k-1} \;, \quad \frac{G'(x)}{G(x)} = (1-x)\sum_{k=1}^{\infty} \chi_k x^{k-1} \quad ,$$

and

(11) $$\psi_k = \chi_k + 0(k) \quad .$$

This suggests a possible relation between the coefficients of $F(x)$ and those of $G(x)$. Note that if

$$\frac{F'(x)}{F(x)} = (1-x)f(x) \quad ,$$

then

$$F(x) = \exp \int_0^x (1-t)f(t)dt \quad .$$

Therefore the following lemma shows how relations (10) and (11) might be applied to our problem:

Lemma 1. Let

$$A(x) = \exp \int_0^x (1-t)a(t)dt \quad ,$$

$$B(x) = \exp \int_0^x (1-t)b(t)dt \quad ,$$

where

$$A(x) = \quad A_k x^k, \; a(x) = \sum a_k x^{k-1}, \; B(x) = \sum B_k x^k, \; b(x) = \sum b_k x^{k-1} \; .$$

Assume the coefficients of $A(x)$ and of $b(x)$ are non-negative and non-decreasing. Then if $a_k \le b_k$ for all k, $A_k \le B_k$; if $a_k \ge b_k$ for all k, $A_k \ge B_k$.

Proof: $A_0 = B_0 = 1$. Assume $a_k \le b_k$ for all k, and $A_k \le B_k$ for $0 \leq k < n$. Then since $A'(x) = (1-x)a(x)A(x)$, we have

$$\begin{aligned} nA_n &= a_n A_o + a_{n-1}(A_1 - A_o) + \ldots + a_1(A_{n-1} - A_{n-2}) \\ &\le b_n A_o + b_{n-1}(A_1 - A_o) + \ldots + b_1(A_{n-1} - A_{n-2}) \\ &= A_o(B_n - b_{n-1}) + A_1(b_{n-1} - b_{n-2}) + \ldots + A_{n-1}b_1 \\ &\le B_o(b_n - b_{n-1}) + B_1(b_{n-1} - b_{n-2}) + \ldots + B_{n-1}b_1 = nB_n \quad . \end{aligned}$$

Essentially the same argument works if $a_k \geq b_k$ for all k.

The problem is now one of estimating the coefficients of

$$G(x) = e^{\frac{1}{\ln 4}\ln^2(1-x)} \quad .$$

Theorem 2. If

$$e^{a \ln^2(1-x)} = \sum c_n x^n \quad , \tag{12}$$

we have

$$c_n = a \ln^2 n + 0((\ln n)(\ln \ln n)) \quad . \tag{13}$$

Proof: First we show that

$$\ln^m(1-x) = \sum_{n=m}^{\infty} \frac{m}{n} H_{m,n} x^n \quad , \tag{14}$$

where

$$H_{m,n} = \sum \frac{1}{a_1 \cdots a_{m-1}}$$

summed over all integers $a_1, \ldots, a_{m-1}$ such that $1 \leq a_i < n$, and the a_i are distinct. This follows inductively, since the derivative of (14) is

$$\frac{\ln^{m-1}(1-x)}{(x-1)} = \sum_{n=m}^{\infty} H_{m,n} x^{n-1} \quad ,$$

and we have

(15) $$H_{m,n} = H_{m,n-1} + \frac{m-1}{n-1} H_{m-1,n-1} .$$

Turning to equation (12), we have

(16) $$\sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} \frac{a^m \ln^{2m}(1-x)}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^m}{m!} \left(\frac{2m}{n}\right) H_{2m,n} \quad x^n .$$

(We define $H_{m,n} = 0$ if $m > n$, so the parenthesized summation is actually a finite sum for any fixed value of n.)

Our theorem relies on the estimates

(17) $$(h_{n-1} - h_{m-1})^{m-1} \le H_{m,n} \le h_{n-1}^{m-1}, \quad \text{if } m \le n .$$

The righthand inequality is obvious, since this is the sum

$$\sum \frac{1}{a_1 \cdots a_{m-1}}$$

without the restriction that the a's are distinct. On the other hand, given any term of

$$(h_{n-1} - h_{m-1})^{m-1} = \sum_{m \le a_1 \le n} \frac{1}{a_1 \cdots a_{m-1}} ,$$

we form a term

$$\frac{1}{b_1 \cdots b_{m-1}}$$

belonging to $H_{m,n}$, where $b_k = a_k - r$ if a_k is the r-th largest of $\{a_1, \ldots, a_{m-1}\}$. Thus, we decrease the largest element by 1, the second largest by 2, and so on; in case of ties, an arbitrary order is taken. No two terms

$$\frac{1}{a_1 \cdots a_{m-1}}$$

map into the same

$$\frac{1}{b_1\cdots b_{m-1}}\,, \quad \text{and} \quad \frac{1}{a_1\cdots a_{m-1}} \le \frac{1}{b_1\cdots b_{m-1}}\,,$$

so the lefthand side of (17) is established.

Putting the righthand side of (17) into (16), we obtain

$$(18) \qquad c_n = \frac{2}{n}\sum_{m=0}^{\infty} \frac{a^m}{(m-1)!} H_{2m,n} \le \frac{2ah_{n-1}}{n}\sum_{m=1}^{\infty} \frac{a^{m-1}h_{n-1}^{2m-2}}{(m-1)!} = \frac{2a}{n}e^{ah_{n-1}^2}$$

On the other hand,

$$(19) \qquad c_n > \frac{2}{n}\,\frac{a^m}{(m-1)!}\,H_{2m,n}$$

for any particular value of m. We choose m to be approximately $ah_{n-1}^2 + 1$, assuming n is large. Then we evaluate the logarithm of the term on the right, using Stirling's approximation and the left hand side of (17), and discarding terms of order less than $(\ln n)(\ln \ln n)$:

$$\ln c_n > \ln\left(\frac{2a}{n}\,\frac{a^{m-1}}{(m-1)!}(h_{n-1} - h_{2m-1})^{2m-1}\right)$$

$$= ah_{n-1}^2 \ln a + 2ah_{n-1}^2 \ln(h_{n-1} - h_{2m-1}) - ah_{n-1}^2(\ln(ah_{n-1}^2) - 1) + 0(\ln n)$$

$$= a\,h_{n-1}^2 + 2a\,h_{n-1}^2 \ln\left(1 - \frac{h_{2m-1}}{h_{n-1}}\right) + 0(\ln n)$$

$$= a\,h_{n-1}^2 - 2a\,h_{n-1}h_{2m-1} + 0(\ln n) \quad .$$

This together with (18) establishes theorem 2.

Theorem 3. Let c_n be as in theorem 2. Then

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1 \quad .$$

Proof: Since $H_{m,n+1} \ge H_{m,n}$, we have

$$\frac{c_{n+1}}{c_n} \ge \frac{n}{n+1}$$

by (16).

We also observe that $H_{m,n} \le h_{n-1} H_{m-1,n}$ and hence by (15)

$$H_{m,n+1} \le H_{m,n} + \frac{m-1}{n} h_{n-1} H_{m-2,n} \quad ;$$

thus

$$c_{n+1} \le \sum_{m=1}^{\infty} \frac{a^m}{m!} \left(\frac{2m}{n+1}\right) H_{2m,n} + \frac{2a}{(n+1)} h_{n-1} \sum_{m=2}^{\infty} \frac{a^{m-1}}{(m-1)!} \left(\frac{2m-1}{2m-2}\right)\left(\frac{2(m-1)}{n}\right) H_{2(m-1),n}$$

$$\le \frac{n}{n+1} c_n + \frac{3ah_{n-1}}{n+1} c_n \quad .$$

<u>Corollary 3.</u> If $P(x)$ is any polynomial, and if

$$\sum C_n x^n = e^{a \ln^2(1-x) + P(x)} \quad ,$$

<u>then</u>

$$\ln C_n = \ln c_n + 0(1) \quad .$$

<u>Proof:</u> If $e^{P(x)} = a_0 + a_1 x + a_2 x^2 + \ldots$, we have

$$\frac{C_n}{c_n} = \frac{a_0 c_n + a_1 c_{n-1} + \ldots + a_n c_0}{c_n} \rightarrow e^{P(1)} \quad .$$

<u>Theorem 4.</u> $\ln \phi_n \sim \frac{1}{\ln 4} (\ln n)^2$.

<u>Proof:</u> Let $\epsilon > 0$ be given. By (11), we can find N so that when $n > N$, $(1-\epsilon)\chi_k < \psi_k < (1+\epsilon)\chi_k$. Apply lemma 1 with $A(x) = F(x)$,

$$b(x) = \psi_1 + \psi_2 x + \ldots + \psi_N x^{n-1} + \sum_{k=N+1}^{\infty} (1+\epsilon)\chi_k x^{k-1} \quad .$$

We find $\phi_n \le C_n$ where, by Corollary 3,

$$\ln C_n \sim \left(\frac{1+\epsilon}{\ln 4}\right) \ln^2 n \quad .$$

Then apply lemma 1 with

$$A(x) = F(x), \quad b(x) = \sum_{k=N+1}^{\infty} (1-\epsilon)\chi_k x^{k-1} \quad .$$

This gives us $\phi_n \geq C'_n$ where

$$\ln C'_n \sim \left(\frac{1-\epsilon}{\ln 4}\right)\ln^2 n \quad .$$

Therefore

$$\left| \frac{\ln \phi_n}{(\ln n)^2} - \frac{1}{\ln 4} \right|$$

is arbitrarily small when n is large enough.

Of course, the estimate we have derived in this theorem is very crude as far as the actual value of ϕ_n is concerned. Empirical tests based on the exact values of ϕ_n for $n \leq 10000$ reveal excellent agreement with the following formula:

$$(20) \qquad \ln \phi_n \approx \frac{\ln n}{\ln 4}(\ln n - 2(\ln \ln n) + 1) + \ln n - .843 \quad .$$

The error is less than .05 for $n > 10$; it reaches a low of about -.05 when n is near 50, then increases to approximately .032 when n is near 5000, and it slowly decreases after that. Thus we can use (20) to calculate

$$(21) \qquad \phi_n \approx .472 n^{1.721} \left(\frac{\sqrt{n}}{\ln n}\right)^{\log_2 n}$$

with an error of at most 5% when $10 < n \leq 10000$. Although formula (20) gives very good accuracy, it should be remembered that only the first term of the expansion has been verified, and the comparatively small values of $\ln \ln n$ for the range of n considered makes it possible that (20) is not the true asymptotic result. On the assumption that the true formula is a relatively "simple" one, however, equation (20) gives striking agreement. A similar situation exists in the study of the partition function; the methods used here can be applied with ease to that problem, to give

$$\log p(n) \sim \pi \sqrt{\frac{2}{3} n} \quad ;$$

the actual asymptotic formula for $p(n)$ itself is

$$p(n) = \left(\frac{1}{4\sqrt{3}} - \frac{1}{4\pi\sqrt{2}(n - \frac{1}{24})}\right) \frac{e^{\pi\sqrt{\frac{2}{3}n - \frac{1}{36}}}}{(n - \frac{1}{24})} + 0(e^{A\sqrt{n}}) ,$$

where $A < \pi\sqrt{\frac{2}{3}}$;

$$p(n) \sim \frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{\frac{2}{3}n}} .$$

It is doubtful that it would have been guessed empirically in either of these forms. For an account of this and a bibliography, see [1] .

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