GENERALIZED BASES FOR THE REAL NUMBERS

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Throughout this paper $\{r_i\}_1^\infty$ will denote a non-increasing real number sequence with limit zero; each of $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ denotes a non-negative integer sequence

$$S = \sum_1^\infty k_i r_i \quad \text{and} \quad S^* = \sum_1^\infty m_i r_i$$

(finite or infinite). We shall consider the possibility of expressing each number x in the interval $(-S^*, S)$ in the form

$$x = \sum_1^\infty a_i r_i$$

where each a_i is an integer satisfying $-m_i \le a_i \le k_i$.

In the classical n-scale number representation, each x in $[0,1]$ can be expressed in the above form, where $n > 1$, and $r_i = n^{-i}$, $k_i = n - 1$, and $m_i = 0$ for each i. Previous generalizations ([6] and [8]) have considered only the expansion of positive numbers with certain restrictions on the coefficient bounds $\{k_i\}_1^\infty$.

In this note we shall extend the previous work to include negative number representations as well as relaxing the restrictions on the coefficients $\{a_i\}_1^\infty$. We shall also consider the question of uniqueness of such representations and the expansion of real numbers using a base sequence $\{\pm r_i\}_1^\infty$ of both positive and negative terms.

DEFINITION. The sequence $\{r_i\}_1^\infty$ is a $\{k, m\}$-base for the interval $(-S^*, S)$ if for each x in $(-S^*, S)$ there is an integer sequence $\{a_i\}_1^\infty$ such that

$$x = \sum_1^\infty a_i r_i\,, \quad \text{and} \quad -m_i \le a_i \le k_i \quad \text{for each } i\,. \tag{1}$$

Our main purpose is to develop an explicit characterization of a $\{k, m\}$-base; to this end we first consider the case where $m_i = 0$ for each i; i.e., a $\{k, 0\}$-base.

LEMMA. The sequence $\{r_i\}_1^\infty$ is a $\{k, 0\}$-base for the interval $(0, S)$ if and only if

$$(2) \qquad r_n \le \sum_{n+1}^{\infty} k_i r_i \qquad \text{for each } n.$$

Proof. If (2) does not hold and

$$r_n > x > \sum_{n+1}^{\infty} k_i r_i \quad ,$$

for some n, it is easily seen that x cannot be expressed in the form (1).

Assume that (2) holds and let x be in $(0, S)$, the conclusion being trivial for $x = 0$. Let $i(1)$ be the least positive integer such that $r_{i(1)} \le x$, and choose $a_{i(1)}$ to be the greatest integer such that $a_{i(1)} \le k_{i(1)}$ and $a_{i(1)} r_{i(1)} \le x$.

If $a_{i(1)} r_{i(1)} < x$, we continue inductively:
Let $i(n)$ be the least positive integer such that

$$(3) \qquad r_{i(n)} \le x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \quad \text{and} \quad i(n) > i(n-1) \quad ;$$

Choose $a_{i(n)}$ to be the greatest integer such that $a_{i(n)} \le k_{i(n)}$ and

$$(4) \qquad a_{i(n)} r_{i(n)} \le x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \quad .$$

In case equality does not hold in (4) for any n, we assert that

$$\sum_{p=1}^{\infty} a_{i(p)} r_{i(p)} = x \quad . \tag{5}$$

Suppose, to the contrary, that for some positive ϵ

$$\sum_{p=1}^{n} a_{i(p)} r_{i(p)} \leq x - \epsilon , \quad \text{for each } n \quad .$$

If $r_{i(n)} < \epsilon$ it follows that

$$(a_{i(n)} + 1) r_{i(n)} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \quad . \tag{6}$$

By the choice of $a_{i(n)}$ this implies that $a_{i(n)} = k_{i(n)}$; furthermore, (6) also yields

$$r_{i(n)+1} \leq r_{i(n)} \leq x - \sum_{p=1}^{n} a_{i(p)} r_{i(p)} \quad ,$$

so that $i(n + 1) = i(n) + 1$. Hence,

$$\sum_{p=1}^{\infty} a_{i(p)} r_{i(p)} = \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} + \sum_{p=i(n)}^{\infty} k_p r_p \leq x \quad . \tag{7}$$

Applying (2) to (7) we see that

$$r_{i(n)-1} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \quad . \tag{8}$$

By the choice of $i(n)$, (8) implies that $i(n) - 1 = i(n-1)$, so that (8) can be written as

$$(a_{i(n-1)} + 1)r_{i(n-1)} \le x - \sum_{p=1}^{n-2} a_{i(p)}r_{i(p)} \quad ,$$

whence $a_{i(n-1)} = k_{i(n-1)}$. Thus it is readily seen that for every n, $i(n) = n$ and $a_{i(n)} = k_n$, which contradicts $x < S$; this establishes (5) and completes the proof.

REMARK. From this Lemma the following is clear:
If $\{r_i\}_1^\infty$ is a $\{k, 0\}$-base for $[0, S)$ and N is a positive integer, then $\{r_i\}_N^\infty$ is a $\{k, 0\}_N^\infty$-base for the interval

$$\left[0, \sum_N^\infty k_i r_i\right) \quad .$$

Theorem 1. The sequence $\{r_i\}_1^\infty$ is a $\{k, m\}$-base for $(-S^*, S)$ if and only if

$$r_n \le \sum_{i=n+1}^\infty (k_i + m_i)r_i \quad \text{for each } n \; . \tag{9}$$

Proof. If (9) does not hold and

$$r_n > x > \sum_{n+1}^\infty (k_i + m_i)r_i \quad ,$$

it follows easily from the Lemma that $x - S^*$ is in $(-S^*, S)$ but $x - S^*$ cannot be expressed as in (1).

To show the sufficiency of (9) we first consider the case where S^* is finite. Let x be in $(-S^*, S)$. By the Lemma, (9) guarantees a sequence $\{a_i\}_1^\infty$

such that

$$x + S^* = \sum_1^\infty a_i r_i , \quad \text{and} \quad 0 \le a_i \le k_i + m_i \quad \text{for each } i .$$

Letting $b_i = a_i - m_i$, we have

$$x = \sum_1^\infty b_i r_i , \quad \text{and} \quad -m_i \le b_i \le k_i \quad \text{for each } i .$$

The case in which S is finite is proved similarly. If both S^* and S are infinite it follows immediately from the Lemma that every non-negative x can be expressed as

$$\sum_1^\infty a_i r_i \quad ,$$

where $0 \le a_i \le k_i$, and every negative x can be so expressed with $-m_i \le a_i \le 0$.

We now wish to establish conditions under which the representations in the form (1) are unique. Since the common decimal expansion is not unique, and this is the special case where $r_i = 10^{-i}$, $m_i = 0$, and $k_i = 9$, we cannot hope for total uniqueness in any non-trivial case. Therefore we adopt a convention similar to that used in identifying the decimal .0999··· with .1000···, viz., we disallow a representation in which $a_i = k_i$ for every i greater than some n. Note that in the proof of the Lemma such representations were not necessary. (This is also the reason that we did not consider the closed interval $[0, S]$ even when S was finite.

Theorem 2. The sequence $\{r_i\}_1^\infty$ yields exactly one $\{k, m\}$-base representation of each x in $(-S^*, S)$ if and only if

$$(10) \qquad r_n = \sum_{i=n+1}^\infty (k_i + m_i) r_i \qquad \text{for each } n .$$

Proof. The sufficiency of (10) is fairly straightforward. Conversely, it is easily seen that for unique representation it is necessary that S^* (and S) be finite. Suppose that S^* is finite and $\{r_i\}_1^\infty$ satisfies (9) but not (10). Then there exists an integer n and a number x such that

$$r_n < x < \sum_{n+1}^{\infty} (k_i + m_i)r_i .$$

Using the construction in the proof of the Lemma, we get a sequence $\{a_i\}_1^\infty$ satisfying

$$x = \sum_1^{\infty} a_i r_i ,$$

and $0 \le a_i \le k_i + m_i$; moreover, since $r_n < x$, at least one of $a_1, \cdots, a_n$ is non-zero. Taking $b_i = a_i - m_i$, we have

$$x - S^* = \sum_1^{\infty} b_i r_i , \qquad \text{where } -m_i \le b_i \le k_i , \tag{11}$$

and for some $i \le n$, $b_i \neq -m_i$.

On the other hand $\{r_i\}_{n+1}^\infty$ is a $\{k + m, 0\}_{n+1}^\infty$-base for the interval

$$\left[0, \sum_{n+1}^{\infty} (k_i + m_i)r_i\right) ,$$

by the Remark following the Lemma. This yields a second $\{k, m\}$-base representation: $x - s = \sum_1 d_i r_i$, where $d_i = -m_i$ for <u>all</u> $i \le n$.

COROLLARY. The sequence $\{r_i\}_1^\infty$ yields a unique $\{k, m\}$-base representation of each x in $(-S^*, S)$ if and only if

$$r_n = (S + S^*)\Big/\Pi_{i=1}^{n}(1 + k_i + m_i) \qquad \text{for each } n.$$

Proof. This is straightforward induction using Theorem 2.

The foregoing theory can be used to consider representations of real numbers in which the base sequence $\{r_i\}_1^\infty$ takes on both positive and negative values. Let A and B be disjoint sets whose union is the set of positive integers, and let C_A and C_B denote their respective characteristic functions. We shall use

$$\left\{(-i)^{C_B(i)} r_i\right\}_1^\infty$$

as the base sequence.

Theorem 3. If $\{q_i\}_1^\infty$ is a positive integer sequence, then

$$\left\{(-1)^{C_B(i)} r_i\right\}_1^\infty$$

is a $\{q, 0\}$-base for the interval

$$\left(-\sum_{i\in B} q_i r_i\,,\ \sum_{i\in A} q_i r_i\right)$$

if and only if

$$r_n \le \sum_{n+1}^{\infty} q_i r_i \qquad \text{for each } n. \tag{12}$$

Proof. Let $k_i = C_A(i)q_i$ and $m_i = C_B(i)q_i$, so that $k_i + m_i = q_i$, $\Sigma_{i\in A} q_i r_i = S$, and $\Sigma_{i\in B} q_i r_i = S^*$. Thus by Theorem 1, (12) is equivalent to $\{r_i\}_1^\infty$ being a $\{k, m\}$-base for $(-S^*, S)$. If (12) holds and x is in $(-S^*, S)$, then

$$x = \sum_1^\infty b_i r_i\,, \quad \text{where} \quad -C_B(i)q_i \le b_i \le C_A(i)q_i .$$

Taking $a_i = (-1)^{C_B(i)} b_i$, we have

$$\sum_1^{\infty} a_i \left[(-1)^{C_B(i)} r_i\right] \quad \text{and} \quad 0 \le a_i \le q_i \ . \tag{13}$$

The converse is proved similarly.

REMARK. It is clear that the representations in (13) are unique if and only if equality holds in (12) for each n.

A related problem is that of expressing a given number x in the form

$$x = \sum_1^{\infty} \epsilon_i r_i \ , \quad \text{where} \quad \epsilon_i = 1 \text{ or } -1 \ . \tag{14}$$

The following solution is proved using Theorem 1.

PROPOSITION. If

$$r_n \le \sum_{n+1}^{\infty} r_i \quad \text{for each } n, \text{ and } |x| \le \sum_1^{\infty} r_i \ ,$$

then x can be expressed in the form (14).

The special case of Theorem 1 in which $k_i = 1$ and $m_i = 0$, for all i, is apparently an old result first proved by Kakeya [7] (cf. [2]). Generalizations of the n-scale (radix n) representation of positive integers which are analogous to the theory presented here have been developed by Alder [1] and Brown [3-5].

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