GENERALIZED FIBONACCI SEQUENCES ASSOCIATED WITH A GENERALIZED PASCAL TRIANGLE

V.C. Harris and Carolyn C. Styles, San Diego State College & San Diego Mesa College

1. INTRODUCTION

In this paper we introduce the numbers

$$(1.1)\qquad \begin{cases} u_n = u_n(p,q,s) = \displaystyle\sum_{i=0}^{\left[\frac{n}{p+sq}\right]} \binom{\left[\frac{n-ip}{s}\right]}{i\ q} & n = 1,2,\cdots \\ u_0 = u_0(p,q,s) = 1 \end{cases}$$

where n, p, q, s are positive integers and $[x]$ is the largest integer in x. The characteristic equation and a generating function are developed and the relation to a generalized Pascal's triangle is exhibited in Section 2. An interesting feature is the repetition of each term g times where $g = (p,s)$. Certain sums and some properties relating to congruence are established in Sections 3 and 4.

The numbers corresponding to the case $s = 1$ are developed in our previous paper [2]. Thus the Fibonacci numbers are those for $p = q = s = 1$. The numbers in Dickinson [1] are the special case $p = a$, $q = 1$, $s = c - a$. By multiplying the binomial coefficients

$$\binom{\left[\frac{n-ip}{s}\right]}{i\ q}$$

by $a^{n-iq}b^{iq}$* before summing, the numbers could be generalized further.

*A more appropriate choice of exponents, suggested by Dr. David Zeitlin, appears in a paper by him which will follow.

2. THE CHARACTERISTIC EQUATION AND GENERATING FUNCTION

We note that

$$\binom{\left[\frac{n+s-ip}{s}\right]}{i\quad q} - \binom{\left[\frac{n-ip}{s}\right]}{i\quad q} = \binom{\left[\frac{n-ip}{s}\right]}{iq-1}$$

or zero, from properties of binomial coefficients. Hence, if $Ef(x) = f(x+1)$ we have $(E^s - 1)u_n$ is a sum of binomial coefficients with first coefficient involving $iq - 1$. After repeating q times there results

$$(E^s - 1)^q u_n(p,q,s) = u_{n-p}(p,q,s), \; n - p \geq 0$$

or

$$(2.1) \quad u_{n+p+qs} = \binom{q}{1} u_{n+p+(q-1)s} - \binom{q}{2} u_{n+p+(q-2)s} + \cdots + (-1)^{q+1} u_{n+p} + u_n$$

Hence the characteristic equation is

$$(2.2) \qquad x^p(x^s - 1)^q - 1 = 0$$

with initial conditions

$$(2.3) \qquad u_0 = u_1 = \cdots = u_{p+qs-1} = 1 \; .$$

It may be remarked that $u_{p+qs} = 2$.

Suppose the arithmetic triangle to be written but with each row repeated s times. Then one sees that $u_n(p,q,s)$ is the sum of the term in the first column and n^{th} row (counting the top row as the zeroth row) and the terms obtained by starting from this term and taking steps p, q — that is, p units up and q units to the right.

When $(p,s) = g > 1$ the sequences $\{u_{ng}\}, \{u_{ng+1}\}, \cdots, \{u_{ng+(g-1)}\}$ are the same since each sequence is determined by the same recursion formula and the same initial conditions.

Let $f(x) = x^p(x^s - 1)^q - 1$ so that $f'(x) = x^{p-1}(x^s - 1)^{q-1}[(p + qs)x^s - p]$. The roots of $f'(x) = 0$ are the roots of

$$x = 0, \quad x^s = 1 \quad \text{and} \quad x^s = \frac{p}{p + qs} \quad .$$

None of the roots of $f'(x) = 0$ is a root of $f(x) = 0$ and $f(x)$ has no multiple root. If the $p + sq$ roots of $f(x)$ are $x_1, x_2, x_3, \cdots, x_{p+sq}$ then the determinant of the coefficients $c_1, c_2, \cdots, c_{p+qs}$ in

$$\sum_{i=1}^{p+qs} c_i x_i^{n+1} = u_n \qquad n = 0, 1, \cdots, p + sq - 1$$

is different from zero. The system can be solved by Cramer's rule using Vandermondians. It results that $c_i = (x_i^s - 1)/[(x_i - 1)\{[p + sq]x_i^s - p\}]$ and hence

$$(2.4) \qquad u_n = \sum_{i=1}^{p+sq} \frac{(x_i^s - 1)x_i^{n+1}}{(x_i - 1)[(p + sq)x_i^s - p]}, \quad n = 0, 1, 2, \cdots$$

To obtain a generating function, write

$$S = \sum_{i=0}^{\infty} u_i x^i \quad .$$

Then by multiplying S by each of

$$(-1)\binom{q}{1}x^s, \ (-1)^2\binom{q}{2}x^{2s}, \cdots, (-1)^q\binom{q}{q}x^{qs} \quad \text{and} \quad -x^{p+qs}$$

and adding, one finds

$$[(1 - x^s)^q - x^{p+sq}]\, S = \sum_{k=0}^{q-1} \sum_{i=0}^{s-1} \sum_{j=0}^{k} (-1)^j \binom{q}{j} x^{ks+i} \quad .$$

Note we have used $u_0 = u_1 = \cdots = u_{p+sq-1} = 1$ and (2.1). Hence

$$\sum_{n=0}^{\infty} u_n x^n = \frac{\sum_{k=0}^{q-1}\sum_{i=0}^{s-1}\sum_{j=0}^{k}(-1)^j\binom{q}{j}x^{ks+i}}{(1-x^s)^q - x^{p+sq}}$$

The numerator is equal to

$$\sum_{i=0}^{s-1} x^i \left\{ \sum_{k=0}^{q-1} (-1)^k \binom{q-1}{k} (x^s)^k \right\} = \sum_{i=0}^{s-1} x^i (1 - x^s)^{q-1}$$

$$= (1 - x^s)^q/(1 - x)$$

Hence

$$\sum_{n=0}^{\infty} u_n x^n = \frac{(1 - x^s)^q/(1 - x)}{(1 - x^s)^q - x^{p+sq}} \tag{2.5}$$

As an example, for $p = 2$, $q = 2$, $s = 3$, this gives

$$\sum_{n=0}^{\infty} u_n(2,2,3)\, x^n = \frac{1 + x + x^2 - x^3 - x^4 - x^5}{1 - 2x^3 + x^6 - x^8}$$

This gives the sequence

$$\{u_n(2,2,3)\} = 1,1,1,1,1,1,1,1,2,2,2,4,4,4,7,7,8,12,12,16,21,21,31,37,\cdots$$

3. SUMS

$$\sum_{i=0}^{n} u_i = \sum_{i=0}^{q-1} \left\{ (-1)^i \binom{q-1}{i} [u_{n+p+s(q-i)} + u_{n+p+s(q-i)-1} + \cdots + u_{n+p+s(q-i)-s+1}] \right\} - s\delta_{1q} \tag{3.1}$$

where δ_{ij} is Kronecker's δ.

This is seen to be true for $q = 1$ and all n by summing $u_0 = u_{p+s} - u_p$; $u_1 = u_{p+s+1} - u_{p+1}; \cdots$; $u_n = u_{p+s+n} - u_{p+n}$. Since $u_0 = u_1 = \cdots = u_{p+sq-1} = 1$, this gives

$$\sum_{i=0}^{n} u_i = \sum_{i=0}^{n+p+s} u_i - (p + s) - \sum_{i=0}^{n+p} u_i + p$$

which is the result. Also this is true for $n = 0$ and all q. We have to show

$$u_0 = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} [u_{p+s(q-i)} + \cdots + u_{p+s(q-i)-(s-1)}]$$

But $u_{p+sq} = 2$ so that by separating the term corresponding to $i = 0$ we get $1 + s(1 - 1)^{q-1} = 1 = u_0$.

It remains to show the result in general. Assume (3.1) to be true for $q \geq 2$ and $n = k$; then

$$\sum_{i=0}^{k+1} u_i = u_{k+1} + \sum_{i=0}^{k} u_i = \sum_{i=0}^{q} (-1)^i \binom{q}{i} u_{k+p+(q-i)s+1} +$$

$$\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} [u_{k+p+s(q-i)} + u_{k+p+s(q-i)-1} + \cdots + u_{k+p+s(q-i)-s+1}]$$

By combining terms

$$u_{k+1+p+s(q-i)} = u_{k+p+s(q-i)+1}$$

using

$$\binom{q}{j} - \binom{q-1}{j-1} = \binom{q-1}{j},$$

the result follows and the theorem is proved.

$$\sum_{i=0}^{n} (-1)^{n-i} u_i = \begin{cases} \displaystyle\sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i\,, & s \text{ even} \\ \displaystyle\frac{1}{1-(-1)^{p+sq}2^q}\left[\sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i + \begin{cases} 0 & p \equiv q \equiv 1 \pmod 2 \\ (-1)^{n+1}\, 2^{q-1} & p \equiv q \equiv 0 \pmod 2 \\ (-1)^{n}\, 2^{q-1} & p \not\equiv q \pmod 2 \end{cases}\right] & s \text{ odd} \end{cases} \tag{3.2}$$

Proof: Solve (2.1) for u_n and write $(-1)^{n-i}u_i$ for $i = n, n-1, \cdots, 0$. The sum of the $(k+1)$st column formed by the expansions is

$$\sum_{i=0}^{n} (-1)^{k+i} \binom{q}{k} u_{n+p+s(q-k)-i} = \sum_{i=0}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)-i+k} \binom{q}{k} u_i - (-1)^{k+n+1}\binom{q}{k}\left[1 - 1 + \cdots + (-1)^{p+s(q-k)-1}\right]$$

since the terms added to obtain the sum on the right have $u_i \equiv 1$ in each case. Hence this gives

$$\sum_{i=0}^{n} (-1)^{k+i} \binom{q}{k} u_{n+p+s(q-k)-i} = \sum_{i=0}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i + (-1)^{n+k}\binom{q}{k} \cdot \epsilon$$

where $\epsilon = 0$, $p + s(q-k) \equiv 0 \pmod 2$ and $= 1$ otherwise.

Summing for $k = 0, 1, \cdots, q$ gives

$$\sum_{i=0}^{n}(-1)^{n-i} u_i = \sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i}\binom{q}{k} u_i$$

$$+ \sum_{k=0}^{q}\sum_{i=0}^{n}(-1)^{n+p+s(q-k)+k-i}\binom{q}{k}u_i +$$

$$\begin{cases} \displaystyle\sum_{k=0}^{q}(-1)^{n+k}\binom{q}{k}\epsilon(k) , & p + s(q - k) \not\equiv 0 \pmod 2 \\ 0 , & p + s(q - k) \equiv 0 \pmod 2 \end{cases}$$

But

$$\sum_{k=0}^{q}\sum_{i=0}^{n}(-1)^{n+p+s(q-k)+k-i}\binom{q}{k}u_i =$$

$$\begin{cases} 0 , & s \equiv 0 \pmod 2 \\ (-1)^{p+sq}2^q \displaystyle\sum_{i=0}^{n}(-1)^{n-i} u_i , & s \equiv 1 \pmod 2 \end{cases}$$

and

$$\sum_{n=0}^{q}(-1)^{n+k}\binom{q}{k}\epsilon(p, q, s, k) =$$

$$\begin{cases} (-1)^{n+1}2^{q-1} , & s \text{ odd } p \equiv q \equiv 0 \pmod 2 \\ (-1)^{n}\ 2^{q-1} , & s \text{ odd } \quad p \not\equiv q \quad \pmod 2 \\ 0 , & \text{otherwise} \end{cases}$$

Combining these results gives the theorem.

It may be remarked that

$$\sum_{i=0}^{n} u_{2i} \text{ and } \sum_{i=0}^{n} u_{2i+1}$$

can be obtained from (3.1) and (3.2).

4. DIVISIBILITY PROPERTIES

Using methods similar to those of our previous paper, one can show the following: Any $p + sq$ consecutive terms are relatively prime. The least nonnegative residues modulo any positive integer m of $u_n(p,q,s)$ are periodic with a period P not exceeding m^{p+sq}. There is no preperiod and each period begins with $p + sq$ terms all unity. Any prime divides infinitely many $u_n(p,q,s)$ since

$$u_{P-p} \equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i} u_{P+s(q-i)} \equiv 0 \pmod{m}.$$

REFERENCES

1. David Dickinson, "On Sums Involving Binomial Coefficients," American Mathematical Monthly, Vol. 57, 1950, pp 82–86.
2. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, 1964, pp 277–289.

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