A POWER IDENTITY FOR SECOND-ORDER RECURRENT SEQUENCES

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1. INTRODUCTION

The following hold for all integers n and k:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n ,$$

$$F^2_{n+k} = (F_k F_{k-1}) F^2_{n+2} + (F_k F_{k-2}) F^2_{n+1} - (F_{k-1} F_{k-2}) F^2_n ,$$

$$F^3_{n+k} = (F_k F_{k-1} F_{k-2}/2) F^3_{n+3} + (F_k F_{k-1} F_{k-3}) F^3_{n+2} - (F_k F_{k-2} F_{k-3}) F^3_{n+1}$$

$$- (F_{k-1} F_{k-2} F_{k-3} /2) F^3_n .$$

These identities suggest that there is a general **expansion** of the form

$$(1.1) \qquad F^p_{n+k} = \sum_{j=0}^{p} a_j (k,p) F^p_{n+j} .$$

Here we show such an expansion does indeed exist, find an expression for the coefficients $a_j(k,p)$, and generalize (1.1) to second order recurrent sequences.

2. A FIBONACCI POWER IDENTITY

Define the Fibonomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1$$

Jarden [4] proved that the term-by-term product z_n of p sequences each obeying the Fibonacci recurrence satisfies

$$(2.1) \qquad \sum_{j=0}^{p+1} (-1)^{j(j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix} z_{n-j}$$

for integral n. In particular, $z_n = F_n^p$ obeys (2.1). Carlitz, [1, Section 1] has shown that the determinant

$$D_p = \left| F^p_{n+r+s} \right| \quad (r, s = 0, 1, \cdots, p)$$

has the value

$$D_p = (-1)^{p(p+1)(n+1)/2} \prod_{j=0}^{p} \binom{p}{j} \cdot (F_1^p F_2^{p-1} \cdots F_p)^2 \neq 0$$

implying that the $p+1$ sequences $\{F_n^p\}, \{F^p_{n+1}\}, \cdots, \{F^p_{n+p}\}$ are linearly independent over the reals. Since each of these sequences obeys the $(p+1)^{th}$ order recurrence relation (2.1), they must span the space of solutions of (2.1). Therefore an expansion of the form (1.1) exists.

To evaluate the coefficients $a_j(k,p)$ in (1.1) we first put $k = 0, 1, \cdots, p$, giving $a_j(k,p) = \delta_{jk}$ for $0 \le j,k \le p$, where δ_{jk} is the Kronecker delta defined by $\delta_{jk} = 0$ if $j \neq k$, $\delta_{kk} = 1$. Next we show that the sequence

$$\{a_j(k,p)\}_{k=0}^{\infty}$$

obeys (2.1) for $j = 0, 1, \cdots, p$. Indeed from (1.1) we find

$$0 = \sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} F^p_{n+k-r}$$

$$= \sum_{j=0}^{p} \left\{ \sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} a_j(k-r,p) \right\} F^p_{n+j} \quad .$$

But the F^p_{n+j} $(j = 0, 1, \cdots, p)$ are linearly independent, so that

$$\sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} a_j(k-r,p) = 0 \quad (j = 0, 1, \cdots, p) \quad .$$

Now consider $b_j(k,p) = (F_kF_{k-1}\cdots F_{k-p})/F_{k-j}(F_jF_{j-1}\cdots F_1F_{-1}\cdots F_{j-p})$ for $j = 0,1,\cdots,p-1$, $b_p(k,p) = \begin{bmatrix} k \\ p \end{bmatrix}$, together with the convention that $F_0/F_0 = 1$. Clearly $b_j(k,p) = \delta_{jk}$ for $0 \le j,k \le p$. Since $\{b_j(k,p)\}_{k=0}^{\infty}$ is the term-by-term product of p Fibonacci sequences, it must obey (2.1). Thus $\{a_j(k,p)\}_{k=0}^{\infty}$ and $\{b_j(k,p)\}_{k=0}^{\infty}$ obey the same $(p+1)^{\text{th}}$ order recurrence relation and have their first $p+1$ values equal $(j = 0,1,\cdots,p)$, so that $a_j(k,p) = b_j(k,p)$. Since $F_{-n} = (-1)^{n+1}F_n$, it follows that

$$F_{-1}\cdots F_{j-p} = F_{p-j}\cdots F_1(-1)^{(p-j)(p-j+3)/2} ,$$

so that for $j = 0,1,\cdots,p-1$, we have

$$a_j(k,p) = (-1)^{(p-j)(p-j+3)/2}\left(\frac{F_kF_{k-1}\cdots F_{k-p+1}}{F_pF_{p-1}\cdots F_1}\right)\left(\frac{F_pF_{p-1}\cdots F_1}{(F_j\cdots F_1)(F_{p-j}\cdots F_1)}\right)\left(\frac{F_{k-p}}{F_{k-j}}\right)$$

$$= (-1)^{(p-j)(p-j+3)/2}\begin{bmatrix} k \\ p \end{bmatrix}\begin{bmatrix} p \\ j \end{bmatrix}(F_{k-p}/F_{k-j}) ,$$

which is also valid for $j = p$ using the convention $F_0/F_0 = 1$. Then (1.1) becomes

$$F^p_{n+k} = \sum_{j=0}^{p}(-1)^{(p-j)(p-j+3)/2}\begin{bmatrix} k \\ p \end{bmatrix}\begin{bmatrix} p \\ j \end{bmatrix}(F_{k-p}/F_{k-j})F^p_{n+j} \tag{2.2}$$

for all k. We remark that since consecutive p^{th} powers of the natural numbers obey

$$\sum_{j=0}^{p+1}(-1)^{p-j}\binom{p+1}{j}(n+j)^p = 0 ,$$

a development similar to the above leads to

$$(n+k)^p = \sum_{j=0}^{p}(-1)^{p-j}\binom{k}{p}\binom{p}{j}\left(\frac{k-p}{k-j}\right)(n+j)^p , \tag{2.3}$$

a result parallel to (2.2)

3. EXTENSION TO SECOND-ORDER RECURRENT SEQUENCES

We now generalize the result of Section 2. Consider the second-order linear recurrence relation

$$y_{n+2} = py_{n+1} - qy_n \quad (q \neq 0) \quad . \tag{3.1}$$

Let a and b be the roots of the auxiliary polynomial $x^2 - px + q$ of (3.1). Let w_n be any sequence satisfying (3.1), and define u_n by $u_n = (a^n - b^n)/(a - b)$ if $a \neq b$, and $u_n = na^{n-1}$ if $a = b$, so that u_n also satisfies (3.1). Following [4], we define the u-generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}_u$ by

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r} \quad (r > 0) \; ; \; \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1 \quad .$$

Jarden [4] has shown that the product x_n of p sequences each obeying (3.1) satisfies the $(p + 1)^{th}$ order recurrence relation

$$\sum_{j=0}^{p+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix}_u x_{n-j} = 0 \quad . \tag{3.2}$$

If all of these sequences are w_n, then it follows that $x_n = w_n^p$ obeys (3.2).

It is our aim to give the corresponding generalization of (1.1) for the sequence w_n; that is, to show there exists coefficients $a_j(k,p,u) = a_j(k)$ such that

$$w_{n+k}^p = \sum_{j=0}^{p} a_j(k)\, w_{n+j}^p \tag{3.3}$$

and to give an explicit form for the $a_j(k)$. Carlitz [1, Section 3] proved that

$$D_p(w) = \left| w_{n+r+s}^p \right| \quad (r,s = 0,1,\cdots,p)$$

is nonzero, showing that the $p + 1$ sequences

$$\{w_n^p\}, \{w_{n+1}^p\}, \cdots, \{w_{n+p}^p\}$$

are linearly independent. Reasoning as before, we see these sequences span the space of solutions of (3.2), so that the expansion (3.3) indeed exists. Putting $k = 0, 1, \cdots, p$ in (3.3) gives $a_j(k) = \delta_{jk}$ for $0 \le j, k \le p$. It also follows as before that the sequence

$$\{a_j(k)\}_{k=0}^{\infty}$$

satisfies (3.2). Now consider

$$b_j(k,p,u) = b_j(k) = u_k u_{k-1} \cdots u_{k-p} / u_{k-j}(u_j u_{j-1} \cdots u_1 u_{-1} \cdots u_{j-p})$$

for $j = 0, 1, \cdots, p - 1$, $b_p(k) = \begin{bmatrix} k \\ p \end{bmatrix}_u$, along with the convention $u_0/u_0 = 1$. Then $b_j(k) = \delta_{jk}$ for $0 \le j, k \le p$. Also $\{b_j(k)\}_{k=0}^{\infty}$ obeys (3.2) because it is the product of p sequences each of which obeys (3.1). Since $\{a_j(k)\}_{k=0}^{\infty}$ and $\{b_j(k)\}_{k=0}^{\infty}$ $(j = 0, 1, \cdots, p)$ obey the same $(p + 1)^{\text{th}}$ order recurrence relation and agree in the first $p + 1$ values, we have $a_j(k) = b_j(k)$. Now $ab = q$, so that $u_{-n} = (a^{-n} - b^{-n})/(a - b) = -q^n u_n$. Then

$$u_{-1} \cdots u_{j-p} = u_{p-j} \cdots u_1 (-1)^{p-j} q^{(p-j)(p-j+1)/2}$$

and thus for $j = 0, 1, \cdots, p - 1$ we see

(3.4)

$$a_j(k) = (-1)^{p-j} q^{(p-j)(p-j+1)/2} \left(\frac{u_k u_{k-1} \cdots u_{k-p+1}}{u_p u_{p-1} \cdots u_1} \right) \left(\frac{u_p u_{p-1} \cdots u_1}{(u_j \cdots u_1)(u_{p-j} \cdots u_1)} \right) \left(\frac{u_{k-p}}{u_{k-j}} \right)$$

$$= (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p}/u_{k-j}),$$

which is also valid for $j = p$ using the convention $u_0/u_0 = 1$. Therefore (3.3) becomes

$$(3.5) \qquad w_{n+k}^p = \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p}/u_{k-j}) w_{n+j}^p,$$

Carlitz has communicated and proved a further extension of this result. Let

$$x_n^{(p)} = w_{n+a_1} w_{n+a_2} \cdots w_{n+a_p} ,$$

where the a_j are arbitrary but fixed nonnegative integers. Then we have

$$(3.6) \qquad x_{n+k}^{(p)} = \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p}/u_{k-j}) x_{n+j}^{(p)} ,$$

where u_0/u_0 still applies. We note that putting $a_1 = a_2 = \cdots = a_p = 0$ reduces (3.6) to (3.5).

To prove (3.6) using previous techniques requires us to show that the sequences

$$\left\{x_n^{(p)}\right\}, \left\{x_{n+1}^{(p)}\right\}, \cdots, \left\{x_{n+p}^{(p)}\right\}$$

are linearly independent. To avoid this, we establish (3.6) by induction on k. Now (3.6) is true for $k = 0$ and all n. Assume it is true for some $k \geq 0$ and all n, and replace n by $n + 1$, giving

$$x_{n+k+1}^{(p)} = \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j \end{bmatrix}_u \frac{u_{k-p}}{u_{k-j}} x_{n+j+1}^{(p)}$$

$$= \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=1}^{p} (-1)^{p-j+1} q^{(p-j+1)(p-j+2)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{u_{k-p}}{u_{k-j+1}} x_{n+j}^{(p)} + \begin{bmatrix} k \\ p \end{bmatrix}_u x_{n+p+1}^{(p)} .$$

It follows from (3.2) that

$$x_{n+p+1}^{(p)} = -\sum_{j=1}^{p+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix}_u x_{n+p+1-j}^{(p)}$$

$$= \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix}_u x_{n+j}^{(p)} .$$

Thus

$$x_{n+k+1}^{(p)} = \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{x_{n+j}^{(p)}}{u_j u_{k-j+1}}$$

$$\cdot\ (u_{p+1}u_{k-j+1} - q^{p-j+1}u_{k-p}u_j).$$

Since

$$u_{p+1}u_{k-j+1} - q^{p-j+1}u_{k-p}u_j = u_{k+1}u_{p-j+1} ,$$

we have

$$x_{n+k+1}^{(p)} = u_{k+1}\begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{u_{p-j+1}}{u_j u_{k-j+1}} x_{n+j}^{(p)}$$

$$= \begin{bmatrix} k+1 \\ p \end{bmatrix} \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j \end{bmatrix}_u \frac{u_{k-p+1}}{u_{k-j+1}} x_{n+j}^{(p)} ,$$

completing the induction step and the proof.

4. SPECIAL CASES

In this section we reduce (3.5) to a general Fibonacci power identity and to an identity involving powers of terms of an arithmetic progression. First if we let $w_n = F_{ns+r}$, $u_n = F_{ns}$, where r and s are fixed integers with $s \neq 0$, then both w_n and u_n satisfy

$$y_{n+2} - L_s y_{n+1} + (-1)^s y_n = 0 . \tag{4.1}$$

The roots of the auxiliary polynomial of (4.1) are distinct for $s \neq 0$, so that w_n and u_n satisfy the conditions of the previous section. In this case the u-generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}_u$ become the s-generalized Fibonacci coefficients $\begin{bmatrix} m \\ t \end{bmatrix}_s$ defined by

$$\begin{bmatrix} m \\ t \end{bmatrix}_s = \frac{F_{ms}F_{(m-1)s}\cdots F_{(m-t+1)s}}{F_{ts}F_{ts-s}\cdots F_s} \quad (t > 0)\,; \begin{bmatrix} m \\ 0 \end{bmatrix}_s = 1 \quad .$$

A recurrence relation for these coefficients is given in [3]. Now here

$$q = (-1)^s, \quad \text{so} \quad (-1)^{p-j} q^{(p-j)(p-j+1)/2} = (-1)^{(p-j)[s(p-j+1)+2]/2} \quad .$$

Then (3.5) yields

$$(4.3) \quad F^p_{(n+k)s+r} = \sum_{j=0}^{p} (-1)^{(p-j)[s(p-j+1)+2]/2} \begin{bmatrix} k \\ p \end{bmatrix}_s \begin{bmatrix} p \\ j \end{bmatrix}_s \frac{F_{(k-p)s}}{F_{(k-j)s}} F^p_{(n+j)s+r} \quad .$$

Putting $s = 1$ and $r = 0$ gives equation (2.2).

On the other hand, if we let $w_n = ns + r$ and $u_n = n$, where r and s are fixed integers, then w_n and u_n obey

$$(4.4) \qquad y_{n+2} - 2y_{n+1} + y_n = 0 \quad .$$

Since the characteristic polynomial of (4.4) has the double root $x = 1$, both w_n and u_n satisfy the conditions for the validity of (3.5). In this case we have $q = 1$ and $\begin{bmatrix} m \\ t \end{bmatrix}_u = \binom{m}{t}$, the usual binomial coefficient. Then (3.5) becomes

$$(4.5) \quad ([n + k]s + r)^p = \sum_{j=0}^{p} (-1)^{p-j} \binom{k}{p}\binom{p}{j}\left(\frac{k - p}{k - j}\right) ([n + j]s + r)^p \quad .$$

This reduces to (2.3) by setting $s = 1$ and $r = 0$.

REFERENCES

1. L. Carlitz, "Some Determinants Containing Powers of Fibonacci Numbers," Fibonacci Quarterly, 4(1966), No. 2, pp 129 – 134.

2. V. E. Hoggatt, Jr. and A. P. Hillman, "The Characteristic Polynomial of the Generalized Shift Matrix," Fibonacci Quarterly, 3(1965), No. 2, pp 91 — 94.

3. Problem H-72, Proposed by V. E. Hoggatt, Jr. Fibonacci Quarterly, 3(1965), No. 4, pp 299 — 300.

4. D. Jarden, "Recurring Sequences," Riveon Lematimatika, Jerusalem (Israel), 1958, pp 42 — 45.

5. Roseanna F. Torretto and J. Allen Fuchs, "Generalized Binomial Coefficients," Fibonacci Quarterly, 2(1964), No. 4, pp 296 — 302.

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