ON A PARTIAL DIFFERENCE EQUATION OF L. CARLITZ

W. Jentsch, University of Halle/S., Germany

Translated by: P.F. Byrd and Monika Aumann, San Jose State College, San Jose, Calif.

SUMMARY

Eine von L. CARLITZ behandelte partielle Differenzengleichung zweiter Ordnung, die mit den FIBONACCI-Zahlen in Beziehung steht, wird mit Hilfe einer algebraisch begründeten, zweidimensionalen Operatorenrechnung gelöst. Die sich hierbei ergebende Lösung ist allgemeiner als diejenige von L. CARLITZ.

In an article [1] by L. Carlitz, a solution of the equation

$$(1) \qquad u_{mn} - u_{m-1,n} - u_{m,n-1} - u_{m-2,n} + 3u_{m-1,n-1} - u_{m,n-2} = 0 \qquad (m,n \geq 2, \text{ integral})$$

was given with the aid of a power series expansion related to the Fibonacci numbers. Although the solution contains only three arbitrary constants (viz., u_{00}, u_{01}, and u_{10}), it is called a "general solution" — a terminology which appears justified only if, besides equation (1), the following secondary conditions, not mentioned in [1], are also imposed:

$$(2) \qquad u_{11} - u_{01} - u_{10} + 3u_{00} = 0 ,$$

$$(3) \qquad u_{0n} - u_{0,n-1} - u_{0,n-2} = 0 \quad \text{for } n \geq 2 ,$$

$$(4) \qquad u_{m0} - u_{m-1,0} - u_{m-2,0} = 0 \quad \text{for } m \geq 2 ,$$

$$(5) \qquad u_{1n} - u_{0n} - u_{1,n-1} + 3u_{0,n-1} - u_{1,n-2} = 0 \quad \text{for } n \geq 2 ,$$

$$(6) \qquad u_{m1} - u_{m-1,1} - u_{m0} - u_{m-2,1} + 3u_{m-1,0} = 0 \quad \text{for } m \geq 2 .$$

The conclusion (1.4) from [1] is valid only under the assumptions (2) to (6). From (2), u_{11} is fixed, and from (3) to (6) the initial values u_{0n}, u_{1n} and u_{m0}, u_{m1} are uniquely determined for $n,m \geq 2$. The general solution of (3), for instance, is

(7) $$u_{0n} = u_{01} F_n + u_{00} F_{n-1} \quad \text{for } n \ge 0,$$

where

(8) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{with } \alpha = \frac{1}{2}(1 + \sqrt{5}),\ \beta = \frac{1}{2}(1 - \sqrt{5}) \quad \text{(n integral).}$$

One can easily verify that the solution (5.4) given by L. Carlitz in [1] reduces to equation (7) for $m = 0$.

The general solution of (1) without secondary conditions thus contains two pairs of arbitrary functions of only one of the two integral variables m and n. We now wish to determine this solution with the aid of the "two-dimensional, discrete operational calculus" developed in [2].

According to the fundamental idea of J. Mikusiński (see perhaps [3]), the usual addition and the two dimensional Cauchy product

(9) $$a_{mn} \cdot b_{mn} = \sum_{\mu,\nu=0}^{m,n} a_{\mu\nu} b_{m-\mu,n-\nu} \quad \text{as multiplication}$$

are introduced in the set of complex-valued functions of two nonnegative integral variables, and the quotient field Q_2 belonging to the integral domain D_2 arising from this means is considered. In order to conform with the relations in [2], we make an index shift in (1) and write

(1') $$D(u_{mn}) = u_{m+2,n+2} - u_{m+1,n+2} - u_{m+2,n+1} - u_{m,n+2} + 3u_{m+1,n+1} - u_{m+2,n} = 0 \qquad (m,n \ge 0).$$

After application of the difference theorem from [2],

$$u_{m+k,n+\ell} = p^k q^\ell u_{mn} - q^\ell \sum_{\kappa=0}^{k-1} p^{k-\kappa} u_{\kappa n} - p^k \sum_{\lambda=0}^{\ell-1} q^{\ell-\lambda} u_{m\lambda} + \sum_{\kappa=0}^{k-1} \sum_{\lambda=0}^{\ell-1} p^{k-\kappa} q^{\ell-\lambda} u_{\kappa\lambda}$$

($u_{mn} \in D_2$; $u_{m\lambda}$, $u_{\kappa n}$ initial values; k, ℓ natural numbers; p, q inverses to shift operators in Q_2), one obtains the operator representation

(10) $$u = \frac{h(p,q,m,n)}{g(p,q)} ,$$

where the numerator is

$$h = \alpha_n h_1 + \gamma_m h_2 + \beta_n h_3 + \delta_m h_4 + \alpha_0 h_5 + \alpha_1 h_6 + \beta_0 h_7 + \beta_1 h_8 ,$$

as one can easily verify with the polynomials

$$h_1(p,q) = p^2q^2 - pq^2 - p^2q + 3pq - p^2 , \quad h_2 = h_1(q,p) ,$$
$$h_3(p,q) = pq^2 - pq - p, \quad h_4 = h_3(q,p), \quad h_5 = -p^2q^2 + pq^2 + p^2q - 3pq,$$
$$h_6(p,q) = -p^2q + pq , \quad h_7 = h_6(q,p) , \quad h_8 = -pq$$

and the coefficients, the given initial values,

(11) $$\alpha_n = u_{0n} , \gamma_m = u_{m0} , \beta_n = u_{1n} , \delta_m = u_{m1} \quad \text{with}$$
$$\alpha_0 = \gamma_0 , \beta_0 = \gamma_1, \alpha_1 = \delta_0 , \beta_1 = \delta_1 .$$

The denominator, a polynomial of degree 4 in p,q is

$$g(p,q) = p^2q^2 - pq^2 - p^2q + 3pq - p^2 - q^2 = g_1(p,q)\, g_2(p,q)$$

with

$$g_1 = pq - \alpha p - \beta q \quad \text{and} \quad g_2 = pq - \beta p - \alpha q ,$$

where α and β have the values given in (8). As can be immediately proved,

$$\frac{h_i}{g(p,q)} \in D_2$$

holds for $i = 1, \cdots, 8$; and these terms are indeed functions of the Fibonacci numbers F_k. If considerations for the operator

$$\frac{p^2q^2}{g(p,q)}$$

are indicated, the calculation for the remaining members of u_{mn} then follows easily. If one conceives

$$\frac{pq^2}{g(p,q)}$$

as a (proper fractional) rational operator of p alone, then there results, by decomposition into partial fractions,

$$\frac{p^2q^2}{g(p,q)} = \frac{1}{\alpha - \beta}\left(\frac{\alpha pq}{g_2(p,q)} - \frac{\beta pq}{g_1(p,q)}\right)\frac{q}{q - 1} \qquad ,$$

and on account of the obvious relations

$$\frac{pq}{g_2(p,q)} = \binom{m+n}{m}\alpha^m\beta^n \ , \qquad \frac{pq}{g_1(p,q)} = \binom{m+n}{m}\alpha^n\beta^m$$

and of the meaning of $q/(q - 1)$ as a "partial summation operator"

$$\frac{q}{q - 1}\, b_{mn} = \sum_{\nu=0}^{n} b_{m\nu} \ ,$$

it follows that

$$\frac{p^2q^2}{g(p,q)} = \frac{1}{\alpha - \beta}\sum_{k=0}^{n}\binom{m+k}{m}\left[\alpha^{m+1}\beta^k - \alpha^k\beta^{m+1}\right]$$

from which, on account of $\alpha^k\beta^k = (-1)^k$ (k integral), of definition (8), of the symmetry of $g(p,q)$, and with the notation G_{mn} for $(p^2q^2)/g$, there finally results

$$(12) \quad \frac{p^2q^2}{g(p,q)} = G_{mn} = G_{nm} = \sum_{k=0}^{n}(-1)^k\binom{m+k}{k}F_{m+1-k} =$$

$$\sum_{k=0}^{m}(-1)^k\binom{n+k}{k}F_{n+1-k} \qquad \text{for } m,n \geq 0.$$

With the aid of (12) the operators h_i/g $(i = 1,\cdots,8)$ can now be immediately represented as functions from D_2. In order to simplify the notation, we define $G_{mn} = 0$ in case an index is negative; according to (12) this is also achieved by stipulating the following:

(13) $$\sum_{k=0}^{\ell} a_k = 0 \ , \quad \binom{\ell}{k} = 0 \qquad \text{for } \ell < 0 \ .$$

(With this agreement, $(1/p^2)\, G_{mn} = G_{m-2,n}$, for example, holds for all $m,n \geq 0$.)

Therewith we obtain, after easy calculation from (10),

(14) $$\begin{aligned} u_{mn} = {} & \alpha_n \cdot (1 + G_{m-2,n}) + \gamma_m \cdot (1 + G_{m,n-2}) \\ & + \beta_n \cdot (G_{m-1,n} - G_{m-1,n-1} - G_{m-1,n-2}) \\ & + \delta_m \cdot (G_{m,n-1} - G_{m-1,n-1} - G_{m-2,n-1}) - \alpha_0 G_{mn} \\ & + (\alpha_0 - \beta_0)G_{m-1,n} + (\alpha_0 - \alpha_1)G_{m,n-1} - (3\alpha_0 - \beta_0 - \alpha_1 + \beta_1)G_{m-1,n-1} \\ & \qquad \text{for all } m,n \geq 0. \end{aligned}$$

(In this the multiplication symbol means multiplication in D_2 and the summand 1 is the identity element of D_2.) If we finally use

$$G_{m-1,n} - G_{m-1,n-1} = (-1)^n \binom{m+n-1}{n} F_{m-n}$$

and correspondingly

$$G_{m,n-1} - G_{m-1,n-1} = (-1)^m \binom{m+n-1}{m} F_{n-m} \quad \text{for } m,n \geq 0 \ ,$$

and carry out the multiplication in D_2 then after simple transformations for $m,n \geq 0$ we obtain from (14)

(15) $$\begin{aligned} u_{mn} = {} & \alpha_n + \gamma_m + \sum_{\nu=0}^{n} G_{m-2,\nu}\alpha_{n-\nu} + \sum_{\mu=0}^{m} G_{\mu,n-2}\gamma_{m-\mu} \\ & + \sum_{\nu=0}^{n} (-1)^{\nu} \binom{m+\nu-1}{\nu} F_{m-\nu}\beta_{n-\nu} - \sum_{\nu=0}^{m} G_{m-1,\nu-2}\beta_{n-\nu} \\ & + \sum_{\mu=0}^{m} (-1)^{\mu} \binom{n+\mu-1}{\mu} F_{n-\mu}\delta_{m-\mu} - \sum_{\mu=0}^{m} G_{\mu-2,n-1}\delta_{m-\mu} \\ & - \alpha_0(-1)^m \binom{m+n}{m} F_{n-m+1} - \beta_0(-1)^n \binom{m+n-1}{n} F_{m-n} \\ & + (\alpha_0 - \alpha_1)(-1)^m \binom{m+n-1}{m} F_{n-m} - (2\alpha_0 + \beta_1)G_{m-1,n-1} \ . \end{aligned}$$

We verify that (15) satisfies equation (1'), however, we only indicate the calculation: to begin with, $D(\alpha_n) = -\alpha_{n+2}$ holds and $D(\gamma_m) = -\gamma_{m+2}$.

Furthermore,

$$D\left(\sum_{\nu=0}^{n} G_{m-2,\nu}\alpha_{n-\nu}\right) = \sum_{\nu=2}^{n+2} \alpha_{n+2-\nu} D(G_{m-2,\nu-2}) + n - \alpha_{n+2}[G_{m0} - G_{m-1,0} - G_{m-2,0}]$$

$$+ \alpha_{n+1}[G_{m1} - G_{m-1,1} - G_{m0} - G_{m-2,1} + 3G_{m-1,0}]$$

$$= \alpha_{n+2} \;;$$

for, it is true that $G_{m0} - G_{m-1,0} - G_{m-2,0} = \begin{cases} 1 & \text{for } m = 0 \,, \\ 0 & \text{for } m > 0 \,, \end{cases}$

$$G_{m1} - G_{m-1,1} - G_{m0} - G_{m-2,1} + 3G_{m-1,0} = 0 \quad \text{for all } m \ge 0$$

and

$$D(G_{m-2,\nu-2}) = 0 \quad \text{for } m \ge 0 \,, \ \nu \ge 2 \,,$$

as one recognizes after some calculation with the aid of (12) and $F_k = (-1)^{k+1}F_{-k}$ (k integral) or as one can read off directly from the fact that G_{mn} in D_2 is inverse to

$$\frac{g}{p^2q^2} = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{q^2} + \frac{3}{pq} - \frac{1}{p^2} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & \cdots \\ -1 & 3 & 0 & 0 & \cdots & \\ -1 & 0 & 0 & \cdots & & \\ 0 & 0 & \cdots & & & \\ \cdots & & & & & \end{pmatrix}$$

by (9).

Analogously one completes the verification. By appropriate calculation one recognizes that the initial conditions (11) are satisfied by (15). Since $\delta_0 = \alpha_1$, and because of definition (13) and of the validity of the relation (3) for F_n, there results for $m = 0$, $n \ge 0$, for instance,

$$u_{0n} = \alpha_n + 0 + G_{0,n-2}\,0 + \binom{n-1}{0} F_n\,0 - \alpha_0 F_{n+1} + (\alpha_0 - \alpha_1)\binom{n-1}{0} F_n = \alpha_n \,.$$

REFERENCES

1. L. Carlitz, A partial difference equation related to the Fibonacci numbers, Fibonacci Quarterly, Vol. 2, No. 3, pp. 185–196 (1964).
2. W. Jentsch, Operatorenrechnung für Funktionen zweier diskreter Variabler, Wiss. Zeitschr. Univ. Halle, XIV, 4, pp. 311–318 (1965).
3. J. Mikusiński, Sur les fondements du calcul opératoire, Studia Math. 11, pp. 41–70 (1950).

* * * * *

RECURRING SEQUENCES

Review of Book by Dov Jarden
By Brother Alfred Brousseau

For some time the volume, Recurring Sequences, by Dov Jarden has been unavailable, but now a printing has been made of a revised version. The new book contains articles published by the author on Fibonacci numbers and related matters in Riveon Lematematika and other publications. A number of these articles were originally in Hebrew and hence unavailable to the general reading public. This volume now enables the reader to become acquainted with this extensive material (some thirty articles) in convenient form.

In addition, there is a list of Fibonacci and Lucas numbers as well as their known factorizations up to the 385th number in each case. Many new results in this section are the work of John Brillhart of the University of San Francisco and the University of California.

There is likewise, a Fibonacci bibliography which has been extended to include articles to the year 1962.

This valuable reference for Fibonacci fanciers is now available through the Fibonacci Association for the price of $6.00. All requests for the volume should be sent to Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif., 94575.

* * * * *

* * * * *