ON A GENERALIZATION OF MULTINOMIAL COEFFICIENTS FOR FIBONACCI SEQUENCES

Eugene E. Kohlbecker, MacMurray College, Jacksonville, Illinois

Let $m = n_1 + n_2 + \cdots + n_k$ be a partition of m into $k \geq 2$ positive integral parts and let $F_0 = 0$, $F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. This is known as the Fibonacci sequence. A multinomial coefficient for the Fibonacci sequence is defined to be the quotient

$$[m; n_1, n_2, \cdots, n_k] = \prod_{j=1}^{m} F_j \Big/ \prod_{j=1}^{n_1} F_j \prod_{j=1}^{n_2} F_j \cdots \prod_{j=1}^{n_k} F_j \; .$$

It is the purpose of this paper to show that such quotients are integer valued. In order to do this we first establish a representation of F_m in terms of a linear combination of the F_{n_i}. This result is of some interest in itself since it contains many of the classic formulae for Fibonacci sequences.

<u>Theorem 1</u>: Let $F_0 = 0$, $F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, and let $m = n_1 + n_2 + \cdots + n_k$ be a partition of m into positive integral parts. Then

$$F_m = \sum_{i=1}^{k} G_i P_i F_{n_i}$$

where $G_1 = 1$, $G_i = F_{n_1+n_2+\cdots n_{i-1}} - 1$, $1 < i \leq k$; and $P_i = \prod_{j=i+1}^{k} F_{n_j+1}$, $1 \leq i < k$, $P_k = 1$.

For the proof of the theorem we require the following Lemmas:

<u>Lemma 1</u>: If $n_1 + n_2 + \cdots + n_k$ and $n_1' + n_2' + \cdots + n_k'$ are partitions of m into $k \geq 2$ positive integral parts where the parts $n_1', n_2', \cdots, n_k'$ are a permutation of the parts $n_1, n_2, \cdots, n_k$, then

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = \sum_{i=1}^{k} G_i' P_i' F_{n_i}'$$

where

$$G_i = F_{n_1+n_2+\cdots+n_{i-1}-1}, \quad 1 < i \le k, \quad G_1 = 1;$$
$$P_i = \prod_{j=i+1}^{k} F_{n_j+1}, \quad 1 \le i < k, \quad P_k = 1;$$
$$G_i' = F_{n_1'+n_2'+\cdots+n_{i-1}'-1}, \quad 1 < i \le k, \quad G_1' = 1;$$
$$P_i' = \prod_{j=i+1}^{k} F_{n_j'+1}, \quad 1 \le i < k, \quad P_k' = 1 .$$

Proof: Since any permutation of the parts $n_1, n_2, \cdots, n_k$ can be obtained by successive transpositions of adjacent parts it suffices to show the conclusion for the case $n_{s+1} = n_s'$ and $n_s = n_{s+1}'$, $n_i = n_i'$ for $i \ne s, s+1$. From the definition of G_i and G_i' we have $G_i = G_i'$ for $1 \le i \le s$ and $s+2 \le i \le k$, $G_{s+1} = F_{n_1+n_2+\cdots+n_{s-1}+n_s-1}$, $G_{s+1}' = F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1}$. We also have $P_i = P_i'$ for $1 \le i \le s-1$ and $s+1 \le i \le k$, $P_s = F_{n_{s+1}+1}P_{s+1}$, and $P_s' = F_{n_s+1}P_{s+1}$. Thus every term in the unprimed sum equals the corresponding term in the primed sum except for the terms where $i = s$ and $i = s+1$. Considering just these terms, we must show that $G_sP_sF_{n_s} + G_{s+1}P_{s+1}F_{n_{s+1}} = G_s'P_s'F_{n_s'} + G_{s+1}'P_{s+1}'F_{n_{s+1}'}$.

$$\begin{aligned}
G_sP_sF_{n_s} + G_{s+1}P_{s+1}F_{n_{s+1}} &= G_sF_{n_{s+1}+1}P_{s+1}F_{n_s} \\
&\qquad + F_{n_1+n_2+\cdots n_{s-1}+n_s-1}P_{s+1}F_{n_{s+1}} \\
&= G_sF_{n_{s+1}+1}F_{n_s} \\
&\qquad + (F_{n_s}F_{n_1+n_2+\cdots n_{s-1}} + G_sF_{n_s-1})F_{n_{s+1}} \\
&= F_{n_s}F_{n_{s+1}}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_s(F_{n_{s+1}+1}F_{n_s} + F_{n_{s+1}}F_{n_s-1}) \\
&= F_{n_s}F_{n_{s+1}}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_sF_{n_s+n_{s+1}} \\
&= F_{n_{s+1}}F_{n_s}F_{n_1+n_2+\cdots n_{s-1}} \\
&\qquad + G_s(F_{n_s+1}F_{n_{s+1}} + F_{n_s}F_{n_{s+1}-1})
\end{aligned}$$

$$= G_s F_{n_s+1} F_{n_{s+1}}$$

$$+ (F_{n_{s+1}} F_{n_1+n_2+\cdots n_{s-1}} + G_s F_{n_{s+1}-1}) F_{n_s}$$

$$= G_s F_{n_s+1} F_{n_{s+1}}$$

$$+ F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1} F_{n_s}$$

$$= G_s F_{n_s+1} P_{s+1} F_{n_{s+1}}$$

$$+ F_{n_1+n_2+\cdots+n_{s-1}+n_{s+1}-1} P_{s+1} F_{n_s}$$

$$= G'_s P'_s F_{n'_s} + G'_{s+1} P'_{s+1} F_{n'_{s+1}} \quad .$$

where we have used repeatedly the classical formula $F_{m+n} = F_{m+1}F_n + F_{n-1}F_m$.

Lemma 2: If $n_1 + n_2 + \cdots + n_k$ is a partition of m into $k \geq 2$ positive integral parts with at least one part (say n_s) greater than 1, then

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = \sum_{i=1}^{k} G'_i P'_i F_{n'_i}$$

where $n_i = n'_i$ for $i \neq s, r$; $n_s - 1 = n'_s$, $n_r + 1 = n'_r$, $s \neq r$, and G_i, P_i, G'_i and P'_i are all defined as in Lemma 1.

Proof: In view of Lemma 1 we can assume that $n_1 > 1$ and show the result for the partitions $n_1 + n_2 + \cdots + n_k$ and $n'_1 + n'_2 + \cdots + n'_k$ where $n'_1 = n_1 - 1$, $n'_2 = n_2 + 1$, $n'_i = n_i$ for $3 \leq i \leq k$. For this choice, $G_i = G'_i$ for $i = 1$ and $3 \leq i \leq k$, $G_2 = F_{n_1-1}, G'_2 = F_{n_1-2}$, and $P_i = P'_i$ for $1 < i \leq k$.

Here every term in the unprimed sum equals the corresponding term in the primed sum except for $i = 1, 2$. Considering only these terms,

$$G_1P_1F_{n_1} + G_2P_2F_{n_2} = (F_{n_1}) \prod_{j=2}^{k} F_{n_j+1} + (F_{n_1-1}F_{n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1}F_{n_2+1} + F_{n_1-1}F_{n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1+n_2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_2+2}F_{n_1-1} + F_{n_2+1}F_{n_1-2}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= (F_{n_1-2}) \prod_{j=2}^{k} F_{n_j+1} + (F_{n_2+2}F_{n_1-1}) \prod_{j=3}^{k} F_{n_j+1}$$

$$= G_1' P_1' F_{n_1'} + G_2' P_2' F_{n_2'}$$

which completes the proof.

We now proceed to the proof of the theorem. When $m = k$ we have $n_i = 1$, $G_1 = 1$, $G_i = F_{i-2}$ for $2 \le i \le k$, $P_i = 1$ for $1 \le i \le k$ and

$$\sum_{i=1}^{k=m} G_i P_i F_{n_i} = F_1 + \sum_{i=0}^{m-2} F_i = F_1 + (F_m - 1) = F_m$$

by a well-known result for the Fibonacci sequence. When $m = k + 1$, all the parts are 1 except one part which is 2. By Lemma 1 we can assume that $n_k = 2$. For this we have $G_i = F_{i-2}$ for $1 < i \le k$, $G_1 = 1$, $P_i = F_2^{k-i-1}F_3$ for $1 \le i < k$, $P_k = 1$. Thus

$$\sum_{i=1}^{k} G_i P_i F_{n_i} = F_3 F_{k-1} + F_{k-2} = F_{k-1} + (F_{k-1} + F_{k-2}) = F_{k+1} \quad .$$

Now assume $m \ge k + 2$ and let $m = n_1 + n_2 + \cdots + n_k$ with $n_1 \le n_2 \le \cdots \le n_k$. There are two cases, $n_k \ge 3$ or $n_k \ge 2$ and $n_{k-1} \ge 2$. By applying Lemma 2 we can reduce the second case to the first. Thus we need only consider $n_1 \le n_2 \le \cdots \le n_k$ with $n_k \ge 3$. We assume that the result is valid for the partitions

$$m - 1 = n_1' + n_2' + \cdots + n_k' \quad \text{where} \quad n_i' = n_i\,,\ 1 \le i < k,\ n_k' = n_k - 1$$

$$m - 2 = n_1'' + n_2'' + \cdots + n_k'' \quad \text{where} \quad n_i'' = n_i\,,\ 1 \le i < k,\ n_k'' = n_k - 2$$

and show it holds for the partition

$$m = n_1 + n_2 + \cdots + n_k .$$

We have $G_i = G_i' = G_i''$ for $1 \le i \le k$ and

$$P_i = (F_{n_k+1}) \prod_{j=i+1}^{k-1} F_{n_j+1} ,$$

$$P_i' = (F_{n_k}) \prod_{j=i+1}^{k-1} F_{n_j+1} ,$$

$$P_i'' = (F_{n_k-1}) \prod_{j=i+1}^{k-1} F_{n_j+1}$$

for $1 \le i < k$, $P_k = P_k' = P_k'' = 1$. Hence,

$$F_m = F_{m-1} + F_{m-2} = \sum_{i=1}^{k} G_i' P_i' F_{n_i'} + \sum_{i=1}^{k} G_i'' P_i'' F_{n_i''}$$

$$= \sum_{i=1}^{k-1} G_i \left(\sum_{j=i+1}^{k-1} F_{n_j+1} \right) (F_{n_k} + F_{n_k-1}) F_{n_i}$$

$$+ G_k P_k (F_{n_k-1} + F_{n_k-2})$$

$$= \sum_{i=1}^{k} G_i P_i F_{n_i} ,$$

which is the desired result.

Utilizing the result of Theorem 1 we prove the following theorem:

Theorem 2: Let m and r be integers, $m \ge r \ge 2$, and let $n_1 + n_2 + \cdots + n_k$ be a partition of m into positive integral parts.

Then $[m;n_1, n_2, \cdots, n_r]$ is an integer.

Proof: If $m = 2$, then $r = 2$, and the only admissible partition has $n_1 = n_2 = 1$. Clearly $[2;1,1]$ is an integer. Now let $m > 2$ and assume that for every partition of $m - 1$ into positive integers where $m - 1 \geq s \geq 2$ we have that $[m - 1; n_1, n_2, \cdots, n_s]$ is an integer. If $r = m$, then each $n_i = 1$, $1 \leq i \leq r$, and $[m;n_1, n_2, \cdots, n_r]$ is an integer. If $2 \leq r < m$ then $m - 1 \geq r \geq 2$, and by the induction hypothesis $[m - 1;n_1 - 1, n_2, \cdots, n_r]$, $[m - 1;n_1, n_2 - 1, \cdots, n_r]$, $\cdots$, $[m - 1; n_1, n_2, \cdots, n_r - 1]$ are all integers.

Now

$$[m;n_1, n_2, \cdots, n_r] = \sum_{i=1}^{k} G_i P_i [m - 1; n_1, \cdots, n_{i-1}, \cdots n_r]$$

where we have used Theorem 1 to write F_m as a linear combination of the F_{n_i}, $1 \leq i \leq r$. The right-hand side is an integer since all the terms are integers. This completes the proof of the theorem.

<u>Editorial Comment:</u> The special multinomial coefficient where $k = 2$, that is, for $m = n_1 + n_2$,

$$\prod_{j=1}^{m} F_j \Bigg/ \prod_{j=1}^{n_1} F_j \prod_{j=1}^{n_2} F_j \quad ,$$

has been given the fitting name, "Fibonomial coefficient." Fibonomial coefficients appeared in this Quarterly in advanced problem H-4, proposed by T. Brennan and solved by J. L. Brown, Oct., 1963, p. 49, and in Brennan's paper, "Fibonacci Powers and Pascal's Triangle in a Matrix," April and October, 1964. Also, a proof of the theorem of this paper for the case $k = 2$ appears in D. Jarden's <u>Recurring Sequences</u>, p. 45.

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