ON THE MORGAN-VOYCE POLYNOMIAL GENERALIZATION OF THE FIRST KIND

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In recent years, a number of papers appeared on the subject of generalization of the Morgan-Voyce (M-) polynomials (see, e.g., André-Jeannin [1]-[3] and Horadam [4]-[7]). The richness of results in these works prompted our investigation on this subject. We further generalized the M-polynomials in a particular way and obtained some new relations by means of the line-sequential formalism developed earlier (see, e.g., [8]-[10]). It was also shown that many known results were obtainable from these relations in a simple and systematic manner.

The recurrence relation of the M-polynomials is given by

$$-m_n+(2+x)m_{n+1}=m_{n+2},\ n\in\mathbf{Z}, \tag{1}$$

where m_n denotes the n^{th} term in the line-sequence; and $c=-1$ and $b=2+x$ are the parametric coefficients with x being the polynomial variable. The pair of basis, see (1.3a) and (1.4a) in [9], is given by, respectively,

$$M_{1,0}(-1,2+x):\ldots,[1,0],-1,-(2+x),-(3+4x+x^2),\ldots, \tag{2a}$$

$$M_{0,1}(-1,2+x):\ldots,[0,1],2+x,3+4x+x^2,4+10x+6x^2+x^3,\ldots, \tag{2b}$$

which spans the two-dimensional M-vector space.

Let the n^{th} element of the line-sequence $M_{i,j}$ be denoted by $m_n[i,j]$, then by the definition of translation operation, (3.1) in [8], we have

$$Tm_n[1,0]=m_{n+1}[1,0];\ \ Tm_n[0,1]=m_{n+1}[0,1]. \tag{3a}$$

From (2a) and (2b), obviously the following translational relations hold:

$$TM_{1,0}=-M_{0,1},\ \ TM_{0,1}=M_{1,2+x}, \tag{3b}$$

where we have applied the rule of scalar multiplication in [8]. The first relation above also states the translational relation between the two basis line-sequences. In terms of the elements, it takes the form

$$m_{n+1}[1,0]=-m_n[0,1], \tag{3c}$$

in agreement with formula (1.2b) in [10]. Also, by the parity relations (1.3a) and (1.3b) in [10], we have, between the elements in the positive and negative branches of each of the two basis line-sequences, the following relations, respectively,

$$m_{-n}[1,0]=-m_{n+2}[1,0], \tag{4a}$$

$$m_{-n}[0,1]=-m_n[0,1]. \tag{4b}$$

The negative branches in (2a) and (2b) can be obtained by applying these relations, respectively.

Let the generating pair of a line-sequence be $[i,j]$, where $j=i+sx+r$, and $i,s,r\in\mathbf{Z}$, denote a set of parametric constants. This generating pair specifies a corresponding family of line-sequences lying in the M-space. We call this way of generalization adopted by André-Jeannin

[1] the generalization of the first kind, hence the title of this report. Later, André-Jeannin [2] also generalized the recurrence relation (1); thus, from the line-sequential point of view, generalized the M-space itself. We call this latter way of generalization the generalization of the second kind. In this report we shall concern ourselves with the former case only. The latter case will be discussed in a separate report.

Table 1 below gives the line-sequential conversion of those polynomial sequences treated in this report. The parametric coefficients in the Morgan-Voyce line-sequence are implicit in the designation of the letter M and henceforth omitted. There appears in the literature more than one set of conventions, we shall stick to those adopted in this table.

TABLE 1. Line-Sequences and Elements Conversion

Polynomials	Elements	Line-Sequences	References
	$m_n[i, i+sx+r]$	$M_{i,i+sx+r}$	(5*a*), (5*b*)
$B_n(x)$	$m_n[0, 1]$	$M_{0,1}$	[11]
$b_n(x)$	$m_n[1, 1]$	$M_{1,1}$	[11]
$P_n^{(r)}(x)$	$m_n[1, 1+x+r]$	$M_{1,1+x+r}$	[1]
$Q_n^{(r)}(x)$	$m_n[2, 2+x+r]$	$M_{2,2+x+r}$	[4]
$R_n^{(r,u)}(x)$	$m_n[u, u+x+r]$	$M_{u,u+x+r}$	[5]
$U_n(y)$	$m_n[0, 1]$	$M_{0,1}$	[11]

The line-sequence $M_{i,j}$ can be decomposed according to the rules of linear addition and scalar multiplication, see [8], as

$$M_{i,i+sx+r} = M_{1,1+sx+r} + (i-1)M_{1,1}. \tag{5a}$$

In terms of the elements, this becomes

$$m_n[i, i+sx+r] = m_n[1, 1+sx+r] + (i-1)m_n[1, 1]. \tag{5b}$$

Putting $i = u$, $s = 1$, and using the conversion in Table 1, we obtain

$$R_n^{(r,u)}(x) = P_n^{(r)}(x) + (u-1)b_n(x). \tag{5c}$$

This is Theorem 1 in [5] and, equivalently, Theorem 1 in [6]. See the Remark below for further explanation.

We may also decompose $M_{i,j}$ in the following manner:

$$M_{i,i+sx+r} = M_{i,2i+sx} + (r-i)M_{0,1}. \tag{6a}$$

Let $i = s = 1$, then we obtain

$$M_{1,1+x+r} = M_{1,2+x} + (r-1)M_{0,1}. \tag{6b}$$

In terms of the elements, applying (3a) and (3b), we find

$$m_n[1, 1+x+r] = m_{n+1}[0, 1] + (r-1)m_n[0, 1]. \tag{6c}$$

Applying conversions in Table 1, we obtain

$$P_n^{(r)}(x) = B_{n+1}(x) + (r-1)B_n(x) \tag{6d}$$

or, equivalently,

$$R_n^{(r,1)}(x) = U_{n+1}\left(\frac{2+x}{2}\right) + (r-1)U_n\left(\frac{2+x}{2}\right), \tag{6e}$$

which is (4.6) in [5].

If we decompose $M_{i,j}$ in the following manner,

$$M_{i,i+sx+r} = M_{1,2+sx} + (i-2+r)M_{0,1} + (i-1)M_{1,0}, \tag{7a}$$

and let $i = u$ and $s = 1$. Then, using (3b) and (3c), we obtain

$$M_{u,u+x+r} = TM_{0,1} + (u-2+r)M_{0,1} - (u-1)T^{-1}M_{0,1},$$

which, in terms of the elements, becomes

$$m_n[u, u+x+r] = m_{n+1}[0,1] + (u-2+r)m_n[0,1] - (u-1)m_{n-1}[0,1]. \tag{7b}$$

Using the conversions in Table 1, we obtain

$$R_n^{(r,u)}(x) = B_{n+1}(x) + (u-2+r)B_n(x) - (u-1)B_{n-1}(x)$$

or, equivalently,

$$R_n^{(r,u)}(x) = U_{n+1}\left(\frac{2+x}{2}\right) + (u-2+r)U_n\left(\frac{2+x}{2}\right) - (u-1)U_{n-1}\left(\frac{2+x}{2}\right). \tag{7c}$$

This is Theorem 2 in [5] (with a minor typographical correction). It is also valid for negative values of the index n (ref. Theorem 2 in [6]). See the Remark below.

We may also decompose $M_{i,j}$ in the following manner:

$$M_{i,j} = iM_{1,0} + jM_{0,1} \tag{8a}$$

$$= iM_{1,0} + iM_{0,1} + sxM_{0,1} + rM_{0,1}. \tag{8b}$$

Following André-Jeannin [1] and Horadam [4], we define

$$\mathbf{m}_n(i,s,r) = \sum_k \mathbf{m}_{n,k}(i,s,r)x^k, \quad k \geq 0. \tag{8c}$$

where the notation has been modified slightly for typographical convenience but is otherwise easily recognizable as compared to the relating symbols used in [1] and [4]. It is known (see [1]) that the coefficients of x in the basis line-sequence $M_{0,1}$ are generated by the combinatorial function $\binom{n+k}{2k+1}$; by the translational relation (3b), the coefficients in the complementary basis line-sequence $M_{1,0}$ are then generated by $-\binom{n+k-1}{2k+1}$. Substituting into (8b), using Pascal's theorem, we obtain the general coefficient formula:

$$\mathbf{m}_{n,k}(i,s,r) = (i-s)\binom{n+k-1}{2k} + s\binom{n+k}{2k} + r\binom{n+k}{2k+1}. \tag{9}$$

Repeated use of Pascal's theorem leads to relations for some special cases, following are some important examples.

Example 1: Let $i = 1$ and $s = 1$; we obtain formula (9) in [1]:

$$\mathbf{m}_{n,k}(1,1,r) = \binom{n+k}{2k} + r\binom{n+k}{2k+1}. \tag{10}$$

Example 2: Let $i = 2$ and $s = 1$; we obtain Theorem 1 in [4]:

$$\mathbf{m}_{n,k}(2,1,r) = \binom{n+k-1}{2k} + \binom{n+k}{2k} + r\binom{n+k}{2k+1}. \tag{11}$$

Example 3: Let $i = u$ and $s = 1$; we obtain formula (2.12) in [5]:

$$\mathbf{m}_{n,k}(u,1,r) = (u-1)\binom{n+k-1}{2k} + \binom{n+k}{2k} + r\binom{n+k}{2k+1}. \tag{12}$$

Example 4: Applying the "negative whole" formula

$$\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$$

(which has its origin in the reflection symmetry of the Pascal array) to (9), we obtain the equivalent formula for $-n$:

$$\mathbf{m}_{-n,k}(i,s,r) = (i-s)\binom{n+k}{2k} + s\binom{n+k-1}{2k} - r\binom{n+k}{2k+1}. \tag{13}$$

Putting $i = u$ and $s = 1$, we see that it reduces to

$$\mathbf{m}_{-n,k}(u,1,r) = (u-1)\binom{n+k}{2k} + \binom{n+k-1}{2k} - r\binom{n+k}{2k+1}, \tag{14}$$

which is equation (2.9) in [6]. And so forth.

It is easy to verify that

$$\binom{n+k-1}{2k} = 2\binom{n+k-2}{2k} - \binom{n+k-3}{2k} + \binom{n+k-3}{2k-2}. \tag{15}$$

Using this identity, we obtain

$$\mathbf{m}_{n,k}(i,s,r) = 2\mathbf{m}_{n-1,k}(i,s,r) - \mathbf{m}_{n-2,k}(i,s,r) + \mathbf{m}_{n-1,k-1}(i,s,r). \tag{16}$$

This reduces to (7) in [1] if $i = s = 1$; to (2.10) in [4] if $i = 2$ and $s = 1$; and to (2.10) in [5] if $i = u$ and $s = 1$.

Applying the "negative whole" formula to (15), we obtain

$$\binom{-n+k}{2k} = 2\binom{-n+k+1}{2k} - \binom{-n+k+2}{2k} + \binom{-n+k}{2k-2}. \tag{17}$$

Using this identity, we obtain

$$\mathbf{m}_{-n,k}(i,s,r) = 2\mathbf{m}_{-n-1,k}(i,s,r) - \mathbf{m}_{-n-2,k}(i,s,r) + \mathbf{m}_{-n-1,k-1}(i,s,r), \tag{18}$$

which reduces to (2.7) in [6] if $i = u$ and $s = 1$.

Remark: Both identities (9) and (13) and identities (15) and (17) are valid for both positive and negative values of index n, a property intrinsic to the line-sequential formalism. Since (5a) is a line-sequential formulation, this means that equation (5c) is valid irrespective of the positivity or the negativity of the index n. Therefore, equation (5c) is equivalent to both Theorem 1 in [5] and Theorem 1 in [6]. Similarly, equation (7c) is equivalent to Theorem 2 in [5] and also equivalent to Theorem 2 in [6].

There are some special cases that are worth our attention.

Case 1. Let $i=s-r$ in (9). We then have

$$\mathbf{m}_{n,k}(s-r,s,r)=r\binom{n+k-1}{2k+1}+s\binom{n+k}{2k}=ra_{n-2,k}^{(1)}+sa_{n,k}^{(0)}. \tag{19}$$

This translates into the decomposition formula

$$M_{i,j}=(s-r)M_{1,0}+s(1+x)M_{0,1}. \tag{20}$$

The polynomial line-sequence is as follows:

$$\begin{gathered}M_{s-r,s(1+x)}(-1,2+x):\ldots,[s-r,s(1+x)],s+r+3sx+sx^2,\\ s+2r+(6s+r)x+5sx^2+sx^3,\ldots.\end{gathered} \tag{21}$$

Case 2. Let $s=r$ in (9). We then have

$$\mathbf{m}_{n,k}(i,r,r)=(i-r)\binom{n+k-1}{2k}+r\binom{n+k+1}{2k+1}=(i-r)a_{n-1,k}^{(0)}+ra_{n,k}^{(1)}. \tag{22}$$

This translates into the decomposition formula

$$M_{i,j}=iM_{1,0}+(i+r(1+x))M_{0,1}. \tag{23}$$

The polynomial line-sequence is as follows:

$$M_{i,(i+r(1+x))}(-1,1+2x):\ldots,[i,i+r(1+x)],i(1+x)+r(2+3x+x^2),\ldots. \tag{24}$$

Case 3 (a special case of Case 2). Let $i=0$ and $s=r$ in (9). Then we have

$$\mathbf{m}_{n,k}(0,r,r)=r\left(\binom{n+k-1}{2k+1}+\binom{n+k}{2k}\right)=r(a_{n-2,k}^{(1)}+a_{n,k}^{(0)}). \tag{25}$$

This translates into the decomposition formula, from (23),

$$M_{i,j}=r(1+x)M_{0,1}. \tag{26}$$

Hence, this reduces to a multiple of the second basis. The polynomial line-sequence is as follows:

$$M_{0,r(1+x)}(-1,1+2x):\ldots,[0,r(1+x)],r(2+3x+x^2),r(3+7x+5x^2+x^3),\ldots. \tag{27}$$

Case 4. Let $i=2r$ and $s=r$ in (9). We then have

$$\mathbf{m}_{n,k}(2r,r,r)=r\left(\binom{n+k-1}{2k}+\binom{n+k+1}{2k+1}\right)=r(b_{n,k}^{(0)}-a_{n,k}^{(0)}+a_{n,k}^{(1)}), \tag{28}$$

where $b_{n,k}^{(0)}$ is as given in Table 2 below. This translates into the decomposition formula

$$M_{2r,r(3+x)}=r(2M_{1,0}+(3+x)M_{0,1}). \tag{29}$$

The polynomial line-sequence is as follows:

$$M_{2r,\,r(3+x)}(-1, 1+2x): \ldots, [2r, r(3+x)], r+7rx+2rx^2, -2r+8rx+16rx^2+4rx^3, \ldots. \tag{30}$$

Note that, for $-n$, we also have, from (28),

$$\mathbf{m}_{-n,\,k}(2r, r, r) = r\left(\binom{n+k}{2k} + \binom{n+k-1}{2k} - \binom{n+k}{2k+1}\right) = r(2b_{n,k}^{(0)} - b_{n,k}^{(1)}), \tag{31}$$

where $b_{n,k}^{(1)}$ is as given in Table 2 below. And so forth.

Table 2 is a compilation of some conversion relations for convenience of reference.

TABLE 2. Conversion Relations

Relations	References
$\mathbf{m}_{n,k}(i, s, r) = (i-s)a_{n-1,k}^{(0)} + sa_{n,k}^{(0)} + ra_{n-1,k}^{(1)}$	[see(9)]
$\mathbf{m}_{n,k}(1, 1, 0) = \binom{n+k}{2k} = a_{n,k}^{(0)}$	[1]
$\mathbf{m}_{n,k}(1, 1, 1) = \binom{n+k+1}{2k+1} = a_{n,k}^{(1)}$	[1]
$\mathbf{m}_{n,k}(1, 1, r) = \binom{n+k}{2k} + r\binom{n+k}{2k+1} = a_{n,k}^{(r)}$	[1]
$\mathbf{m}_{n,k}(2, 1, 0) = \binom{n+k-1}{2k} + \binom{n+k}{2k} = b_{n,k}^{(0)}$	[4]
$\mathbf{m}_{n,k}(2, 1, 1) = \binom{n+k-1}{2k} + \binom{n+k}{2k} + \binom{n+k}{2k+1} = b_{n,k}^{(1)}$	[4]
$\mathbf{m}_{n,k}(2, 1, r) = \binom{n+k-1}{2k} + \binom{n+k}{2k} + r\binom{n+k}{2k+1} = b_{n,k}^{(r)}$	[4]

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