SOLVING NONHOMOGENEOUS RECURRENCE RELATIONS OF ORDER r BY MATRIX METHODS

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1. INTRODUCTION

Let $a_0, \dots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) be some real or complex numbers. Let $\{C_n\}_{n\geq 0}$ be a sequence of $\mathbb{C}$ (or $\mathbb{R}$). Sometimes, for reasons of convenience, we consider $\{C_n\}_{n\geq 0}$ under its equivalent form as a function $C:\mathbb{N}\to\mathbb{C}$ (or $\mathbb{R}$). And when no possible confusion can arise, we write $C(n)$ rather than C_n and, similarly, in case of an indexed family of functions $C_j:\mathbb{N}\to\mathbb{C}$, we use $C_j(n)$ instead of $C_{j,n}$. Let $\{T_n\}_{n\geq 0}$ be the sequence defined by the following nonhomogeneous recurrence relation of order r,

$$T_{n+1} = a_0T_n + a_1T_{n-1} + \cdots + a_{r-1}T_{n-r+1} + C_{n+1} \quad \text{for } n \geq r-1, \tag{1}$$

where $T_0, \dots, T_{r-1}$ are given initial values (or conditions). In the sequel, we refer to such sequence $\{T_n\}_{n\geq 0}$ as the solution of "recurrence relation (1)." If the function C satisfies

$$C_n = \sum_{j=0}^{d} \beta_j C_{j,n}$$

for some finite sequence of functions $C_0, \dots, C_d:\mathbb{N}\to\mathbb{C}$, the solution $\{T_n\}_{n\geq 0}$ may be expressed as

$$T_n = \sum_{j=0}^{d} \beta_j T_{j,n},$$

where $\{T_{j,n}\}_{n\geq 0}$ is the solution of (1) with $C_n = C_j(n)$. Solutions of (1) have been studied in the case in which C equals a polynomial or a factorial polynomial (see, e.g., [1]-[4], [7], [9], [12]).

The purpose of this paper is to study a matrix formulation of (1), which extends those considered for (1) in [6], [10], and [11], when $C(n) = 0$. This allows us to provide a method for solving equation (1) for a general $C:\mathbb{N}\to\mathbb{C}$. Our expression for general solutions of (1) extends those obtained in [1] for $r \geq 2$. If the nonhomogeneous part equals a polynomial or a factorial polynomial, our general solution allows us to recover a well-known particular solution—Asveld's polynomials and factorial polynomials (see [2], [3], [9]).

This paper is organized as follows. In Section 2 we study an $r \times r$ matrix associated to (1), in connection with r-generalized Fibonacci sequences. In Section 3 we use a matrix formulation with an aim toward solving (1) for arbitrary $C:\mathbb{N}\to\mathbb{C}$. Section 4 is devoted to the study and discussion of our general solution in the polynomial and factorial polynomial cases. Section 5 consists of some final remarks.

2. MATRICES ASSOCIATED TO r-GENERALIZED FIBONACCI SEQUENCES

From the r-generalized Fibonacci sequence $V_{n+1} = a_0V_n + \cdots + a_{r-1}V_{n-r+1}$ for $n \geq 0$, as studied by Andrade and Pethe [1], we take r copies, indexed by s $(0 \leq s \leq r-1)$:

$$V_{n+1}^{(s)} = a_0V_n^{(s)} + \cdots + a_{r-1}V_{n-r+1}^{(s)} \quad \text{for } n \geq 0. \tag{2}$$

We provide these r copies with mutually different sets of initial conditions, that is, $V_{-j}^{(s)} = \delta_{s,j}$ $(0 \leq j \leq r-1,\ 0 \leq s \leq r-1)$, where $\delta_{s,j}$ is the Kronecker symbol. Consider the following $r \times r$ matrix:

$$A = \begin{pmatrix} a_0 & a_1 & & \cdots & a_{r-1} \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Expression (3) shows that the columns and arrows of A are indexed from 0 to $r-1$. The usual matrix indexing form $A = (\alpha_{i,j})_{1\leq i,j\leq r}$ of (3) is given as follows: $\alpha_{1j} = a_{j-1}$ for every $1 \leq j \leq r$, and $\alpha_{ij} = \delta_{i,i-1}$ for every $2 \leq i \leq r$, $1 \leq j \leq r$.

The matrix (3) has been considered for r-generalized Fibonacci sequences in [6], [10], [11].

A straightforward computation allows us to establish that the matrix A is related to the r-generalized Fibonacci sequences (2) as follows.

Proposition 2.1: Let A be the matrix defined by (3). Then, for every $n \geq 0$, we have

$$A^n = (a_{is}^n)_{0\leq i,\,s\leq r-1}$$

where

$$a_{is}^n = V_{n-i}^{(s)}. \tag{4}$$

Remark 2.1: Due to the initial conditions $V_{-j}^{(s)} = \delta_{sj}$ $(0 \leq j \leq r-1,\ 0 \leq s \leq r-1)$, we have indeed that A^0 equals the $r \times r$-identity matrix.

3. SOLVING (1) BY MATRIX METHODS

Consider $X_n = {}^t(T_n, \ldots, T_{n-r+1})$ and $D_n = {}^t(C_n, 0, \ldots, 0)$ for $n \geq r-1$, where ${}^tZ$ denotes the transpose of Z. We can easily verify that (1) is equivalent to the following matrix equation:

$$X_{n+1} = AX_n + D_{n+1}, \quad n \geq r-1, \tag{5}$$

where A is the matrix (3). From (5), we derive that

$$X_n = A^{n-r+1}X_{r-1} + \sum_{k=r}^{n} A^{n-k}D_k, \quad n \geq r. \tag{6}$$

Let $R_n = \sum_{k=r}^{n} A^{n-k}D_k$. Then we can verify that $R_{n+1} = AR_n + D_{n+1}$. From expressions (4), (5), and (6), we derive the following result.

Theorem 3.1: Let $\{T_n\}_{n\geq 0}$ be the solution of (1) whose initial conditions are $T_0, \ldots, T_{r-1}$. Then, for $n \geq 0$, we have

$$T_n = \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1} + \sum_{k=r}^{n} V_{n-k}^{(0)} C_k. \tag{7}$$

Because of (2), the sequence $\{U_n\}_{n\geq 0}$ defined by $U_n = \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1}$ is a solution of the homogeneous part of (1). Thus, the sequence $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$, where

$$W_n^{\langle ps\rangle} = \sum_{k=r}^{n} V_{n-k}^{(0)} C_k = -\sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1} + T_n$$

is a particular solution of (1) that satisfies $W_n^{\langle ps\rangle} = 0$ for $n = 0, 1, \ldots, r-1$. We call $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ the *fundamental particular solution of (1)*. Hence, (6) and Theorem 3.1 allow us to formulate the following result.

Theorem 3.2: Let $\{T_n\}_{n\geq 0}$ be a solution of (1). Then, for $n \geq 0$, we have

$$T_n = T_n^{\langle hs\rangle} + W_n^{\langle ps\rangle} = T_n^{\langle hs\rangle} - \sum_{s=0}^{r-1} V_{n-r+1}^{(s)} T_{r-s-1}^{\langle ps\rangle} + T_n^{\langle ps\rangle}, \tag{8}$$

where $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ is the fundamental particular solution of (1), $\{T_n^{\langle hs\rangle}\}_{n\geq 0}$ is a solution of the homogeneous part of (1) with initial conditions $T_0, \ldots, T_{r-1}$, and $\{T_n^{\langle ps\rangle}\}_{n\geq 0}$ is a particular solution of (1) with initial conditions $T_0^{\langle ps\rangle}, \ldots, T_{r-1}^{\langle ps\rangle}$.

Expression (8) extends the one established in [1], with the aid of Binet's formula in the polynomial case.

4. POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

4.1 Elementary Polynomial Solutions and Asveld's Polynomials

For $C(n) = n^j$ $(0 \leq j \leq d)$, the fundamental particular solution $\{W_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$, called the *elementary fundamental particular solution,* is

$$W_{j,n}^{\langle ps\rangle} = \sum_{q=r}^{n} q^j V_{n-q}^{(0)} \quad \text{for } n \geq r.$$

Let $\{f_n\}_{n\geq r}$ be the sequence of C^∞-functions defined on $\mathbb{R}$ as follows:

$$f_n(x) = \sum_{q=r}^{n} V_{n-q}^{(0)} \exp(qx). \tag{9}$$

For each function f_n, the j^{th} derivative is

$$f_n^{(j)}(x) = \sum_{q=r}^{n} q^j V_{n-q}^{(0)} \exp(px).$$

Expressions (2) and (9) imply that $\{f_n^{(j)}\}_{n\geq r}$ satisfies the following nonhomogeneous recurrence relation of order r,

$$f_{n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i f_{n-i}^{(j)}(x) + (n+1)^j \exp[(n+1)x]. \tag{10}$$

For reasons of simplicity, we suppose that $\{V_n^{(0)}\}_{n\geq -r+1}$ has simple characteristic roots. Thus, Binet's formula takes the form $V_n^{(0)} = \sum_{i=0}^{r-1}\alpha_i\lambda_i^n$. We have to distinguish the following exhaustive cases:

1. $\lambda_i \neq 1$ for every i $(0\leq i\leq r-1)$.
2. There exists d $(0\leq d\leq r-1)$ such that $\lambda_d = 1$.

In the sequel, we suppose (without loss of generality) that $\lambda_0 = 1$.

When $\lambda_i \neq 1$ for all i $(0\leq i\leq r-1)$, we consider

$$H_{1,n}(x) = g_1(x)e^{(n+1)x},\ K_{1,n}(x) = \sum_{i=0}^{r-1} v_i(x)\lambda_i^{n-r+1}, \tag{11}$$

where

$$g_1(x) = \sum_{i=0}^{r-1}\frac{\alpha_i}{e^x-\lambda_i},\quad v_i(x) = \frac{\alpha_i e^{rx}}{\lambda_i - e^x}.$$

And if $\lambda_0 = 1$, we set

$$G_n(x) = \alpha_0\sum_{p=r}^{n} e^{px},\ H_{2,n}(x) = g_2(x)e^{(n+1)x},\ K_{2,n}(x) = \sum_{i=1}^{r-1} v_i(x)\lambda_i^{n-r+1}, \tag{12}$$

where

$$g_2(x) = \sum_{i=1}^{r-1}\frac{\alpha_i}{e^x-\lambda_i}.$$

We set $S_n(x) = H_{1,n}(x)$ if $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$ and $S_n(x) = G_n(x) + H_{2,n}(x)$ if $\lambda_0 = 1$.

Because the λ_i's are characteristic roots, we have

$$K_{p,n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i K_{p,n-i}^{(j)}(x)\ \ (p = 1, 2).$$

Then, from (10), we derive that for $j\geq 0$ we have

$$S_{n+1}^{(j)}(x) = \sum_{i=0}^{r-1} a_i S_{n-i}^{(j)}(x) + (n+1)^j\exp[(n+1)x]. \tag{13}$$

As a consequence, we have the following lemma.

Lemma 4.1:

(a) The elementary fundamental particular solution $\{W_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is given by $W_{j,n}^{\langle ps\rangle} = f_n^{(j)}(0)$. More precisely, we have $W_{j,n}^{\langle ps\rangle} = H_{1,n}^{(j)}(0) + K_{1,n}^{(j)}(0)$ if $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$, where $H_{1,n}(x)$ and $K_{1,n}(x)$ are given by (11), and $W_{j,n}^{\langle ps\rangle} = G_n^{(j)}(0) + H_{2,n}^{(j)}(0) + K_{2,n}^{(j)}(0)$ if $\lambda_0 = 1$, where $G_n(x)$, $H_{2,n}(x)$, and $K_{2,n}(x)$ are given by (12).

(b) For $j\geq 0$, the sequence $\{S_n^{(j)}(0)\}_{n\geq 0}$ is a particular solution of (1) for $C(n) = n^j$.

By Leibnitz's formula, we have

$$H_{p,n}^{(j)}(x) = \sum_{i=0}^{j}\left\{\sum_{k=i}^{j}\binom{k}{j}\binom{i}{k} g_p^{(j-k)}(x)\right\} n^i e^{(n+1)x}\ \text{ for } j\geq 0,$$

where $p = 1, 2$. If $\lambda_0 = 1$ is a characteristic root, then we have

$$G_n^{(j)}(0) = \alpha_0 \sum_{p=r}^{n} p^j = \alpha_0 \sum_{p=0}^{n-r} (n-p)^j .$$

It is known that $\sum_{p=0}^{n} p^j = Q_j(n)$, where $Q_j(n)$ is a polynomial of degree $j+1$. Thus, Lemma 4.1 and (13) allow us to derive the following result.

Theorem 4.2: Let $\{T_n\}_{n\geq 0}$ be a solution of (1) with $C(n) = n^j$. Then the elementary polynomial solution $\{P_j(n)\}_{n\geq 0}$ of (1) is given by $P_j(n) = S_n^j(0)$. More precisely, if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$, we have

$$P_j(n) = \sum_{k=0}^{j} \left\{ \sum_{i=k}^{j} \binom{i}{j} \binom{k}{i} g_1^{(j-i)}(0) \right\} n^k, \tag{14}$$

and if $\lambda_0 = 1$ we have

$$P_j(n) = \alpha_0 \sum_{k=0}^{j+1} \mu_k (n-r)^k + \sum_{k=0}^{j} \left\{ \sum_{i=k}^{j} \binom{i}{j} \binom{k}{i} g_2^{(j-i)}(0) \right\} n^k. \tag{15}$$

If $\lambda_0 = 1$, the polynomial (15) may be written as $P_j(n) = \alpha_0 n^{j+1} + \sum_{k=0}^{j} \nu_{j,k} n^k$, where $\nu_{j,k}$ are constants (real or complex numbers).

Theorem 4.2 shows that particular polynomial solutions $P_j(n)$ $(0 \leq j \leq d)$ defined by (14)-(15) are the well-known Asveld's polynomials studied in [2], [4], [9], and [12]. Our method of obtaining $P_j(n)$ $(0 \leq j \leq d)$ is different. For their computation, we use the classic result on $\sum_{j=0}^{n} p^j = Q_j(n)$ and the j^{th} derivative of $H_{p,n}(x)$ $(p = 1, 2)$ given by (11)-(12). The derivative of $H_{p,n}(x)$ $(p = 1, 2)$ can be derived from the following property.

Proposition 4.3: Let $u(x) = \frac{1}{e^x - \lambda}$ with $\lambda \neq 0, 1$ and $x \neq \ln(\lambda)$ if $\lambda > 0$. Then we have

$$u^{(k)}(x) = \frac{T_k(e^x)}{(e^x - \lambda)^{k+1}},$$

where $T_{k+1} = X(X-\lambda)\frac{dT_k}{dX} - (k+1)XT_k$ for $k \geq 0$.

4.2 Elementary Factorial Polynomial Solutions and Asveld's Polynomials

For $C(n) = n^{(j)}$, the elementary fundamental particular solution $\{\widetilde{W}_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ is

$$\widetilde{W}_{j,n}^{\langle ps\rangle} = \sum_{p=r}^{n} p^{(j)} V_{n-p}^{(0)} \quad \text{for all } n \geq r.$$

Instead of (9), let $\{\widetilde{f}_n\}_{n\geq r}$ be the sequence of C^∞-functions on $\mathbb{R}^* = \mathbb{R} - \{0\}$ defined as follows:

$$\widetilde{f}_n(x) = (-1)^j \sum_{k=r}^{n} V_{n-k}^{(0)} x^{-k+j-1}. \tag{16}$$

The q^{th} $(q \geq 0)$ derivative of $h_{j,k}(x) = x^{-k+j-1}$ $(x \neq 0)$ is $h_{j,k}^{(q)}(x) = (-1)^q (k-j+q)^{(q)} x^{-k+j-q-1}$. Hence, the j^{th} derivative of $\widetilde{f}_n$ is

$$\widetilde{f}_n^{(j)}(x) = \sum_{k=r}^{n} k^{(j)} V_{n-k}^{(0)} x^{-k-1}.$$

From (2), we derive that $\{\tilde{f}_n\}_{n\geq r}$ defined by (16) satisfies

$$\tilde{f}_{n+1}^{(j)}(x)=\sum_{i=0}^{r-1}a_i\tilde{f}_{n-i}^{(j)}(x)+(n+1)^{(j)}x^{-n-2}. \tag{17}$$

As in Subsection 4.1, we suppose that $\{V_n^{(0)}\}_{n\geq -r+1}$ has simple characteristic roots. We also consider the following two exhaustive cases: (a) $\lambda_i\neq 1$ for every i $(0\leq i\leq r-1)$; (b) There exists d $(0\leq d\leq r-1)$ such that $\lambda_d=1$. As in Subsection 4.1, we suppose in the second case that $\lambda_0=1$. The case in which $\lambda_d=1$ for some $d\neq 0$ can be derived easily.

When $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$, we set

$$\tilde{H}_{1,n}(x)=\tilde{g}_1(x)h_{j,n}(x),\ \tilde{K}_{1,n}(x)=\sum_{0\leq i\leq r-1}\tilde{v}_i(x)\lambda_i^{n-r+1}, \tag{18}$$

where

$$\tilde{g}_1(x)=(-1)^j\sum_{i=0}^{r-1}\frac{\alpha_i}{1-x\lambda_i},\ \tilde{v}_i(x)=(-1)^j\frac{\alpha_i x^{j-r}}{\lambda_i x-1}.$$

If $\lambda_0=1$, we set

$$\tilde{G}_n(x)=(-1)^j\alpha_0\sum_{k=r}^{n}h_{j,k}(x),\ \tilde{H}_{2,n}(x)=\tilde{g}_2(x)h_{j,n}(x),\ \tilde{K}_{2,n}(x)=\sum_{i=1}^{r-1}\tilde{v}_i(x)\lambda_i^{n-r+1}, \tag{19}$$

where

$$\tilde{g}_2(x)=(-1)^j\sum_{i=1}^{r-1}\frac{\alpha_i}{1-x\lambda_i}.$$

Because the λ_i's are characteristic roots, we have

$$\tilde{K}_{p,n+1}^{(j)}(x)=\sum_{i=0}^{r-1}\alpha_i\tilde{K}_{p,n-i}^{(j)}(x)\ \ (p=1,2).$$

Then from (17) we derive that, for all $j\geq 0$, we have

$$\tilde{S}_{n+1}^{(j)}(x)=\sum_{i=0}^{r-1}a_i\tilde{S}_{n-i}^{(j)}(x)+(n+1)^{(j)}x^{-n-2}, \tag{20}$$

where $\tilde{S}_n(x)=\tilde{H}_{1,n}(x)$ if $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$ and $\tilde{S}_n(x)=\tilde{G}_n(x)+\tilde{H}_{2,n}(x)$ if $\lambda_0=1$.

Therefore, we have the analog of Lemma 4.1 as follows.

Lemma 4.4

(a) The elementary fundamental particular solution $\{\tilde{W}_{j,n}^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is given by $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{f}_n^{(j)}(1)$. More precisely, we have $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{H}_{1,n}^{(j)}(1)+\tilde{K}_{1,n}^{(j)}(1)$ if $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$, where $\tilde{H}_{1,n}(x)$ and $\tilde{K}_{1,n}(x)$ are given by (18), and $\tilde{W}_{j,n}^{\langle ps\rangle}=\tilde{G}_n^{(j)}(1)+\tilde{H}_{2,n}^{(j)}(1)+\tilde{K}_{2,n}^{(j)}(1)$ if $\lambda_0=1$, where $\tilde{G}_n(x)$, $\tilde{H}_{2,n}(x)$, and $\tilde{K}_{2,n}(x)$ are given by (19).

(b) For $j\geq 0$, the sequence $\{\tilde{S}_n^{(j)}(1)\}_{n\geq 0}$ is a particular solution of (1) for $C_n=n^{(j)}$.

By Leibnitz's formula, we have

$$\tilde{H}_{p,n}^{(j)}(x)=\sum_{k=0}^{j}\binom{k}{j}g_p^{(j-k)}(x)h_{j,n}^{(k)}(x)\ \ (p=1,2).$$

Thus,

$$\widetilde{H}_{p,n}^{(j)}(x)=\sum_{k=0}^{j}(-1)^k\binom{k}{j}g_p^{(j-k)}(x)(n-j+k)^{(k)}x^{-n+j-k-1}\quad(p=1,2).$$

Consider the following "binomial theorem for factorial polynomials," which is designated by Asveld [3] as Lemma 1:

$$(x+y)^{(k)}=\sum_{i=0}^{k}\binom{i}{k}x^{(i)}y^{(k-i)}.$$

Then we have

$$\widetilde{H}_{p,n}^{(j)}(1)=\sum_{i=0}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}g_p^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}\quad(p=1,2).$$

Hence, $\widetilde{H}_{p,n}(1)$ $(p=1,2)$ is a factorial polynomial. If $\lambda_0=1$, we have

$$\widetilde{G}_n^{(j)}(1)=\alpha_0\sum_{k=0}^{n-r}(n-k)^{(j)}.$$

Next, we establish that $\widetilde{G}_n^{(j)}(1)$ is a factorial polynomial.

Lemma 4.5: For $j\geq 0$, we have

$$\sum_{k=0}^{n}k^{(j)}=\sum_{k=0}^{j+1}\beta_{j,k}n^{(k)},$$

where $\beta_{j,k}$ are constants (real or complex numbers).

Proof: Consider Stirling numbers of the first kind $s(t,j)$ and Stirling numbers of the second kind $S(t,j)$, which are defined by

$$x^{(j)}=\sum_{t=0}^{j}s(t,j)x^t\quad\text{and}\quad x^i=\sum_{t=0}^{i}S(t,i)x^{(t)}.$$

By successive applications of the two preceding formulas and the following classic result,

$$\sum_{k=0}^{n}k^t=\sum_{i=0}^{t+1}a_{i,t}n^i,$$

we derive that

$$\sum_{k=0}^{n}k^{(j)}=\sum_{q=0}^{j+1}\beta_{j,q}n^{(q)},$$

where

$$\beta_{j,q}=\sum_{i=q}^{j}\sum_{i=0}^{t+1}a_{i,t}s(t,j)S(q,j).\quad\square$$

Now, using Lemma 4.4, we derive the following result.

Theorem 4.6: Let $\{T_n\}_{n\geq 0}$ be a solution of (1) with $C(n)=n^{(j)}$. Then the elementary factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n\geq 0}$ of (1) is given by $\widetilde{P}_j(n)=\widetilde{S}_n^{(j)}(1)$. More precisely, if $\lambda_i\neq 1$ for all i $(0\leq i\leq r-1)$, we have

$$\widetilde{P}_j(n) = \sum_{i=0}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}\widetilde{g}_1^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}. \tag{21}$$

And if $\lambda_0 = 1$, we have

$$\widetilde{P}_j(n) = (-1)^j\alpha_0\sum_{k=0}^{j+1}\gamma_{j,k}n^{(k)} + \sum_{i=1}^{j}\left(\sum_{k=i}^{j}(-1)^k\binom{k}{j}\binom{i}{k}\widetilde{g}_2^{(j-k)}(1)(k-j)^{(k-i)}\right)n^{(i)}, \tag{22}$$

where $\gamma_{j,k}$ are constants (real or complex numbers).

The particular factorial polynomial solutions $\widetilde{P}_j(n)$ $(0 \le j \le d)$ defined by (21)-(22) are the well-known Asveld factorial polynomials studied in [4] and [7]. Our method for obtaining $\widetilde{P}_j(n)$ $(0 \le j \le d)$ is different from Asveld's. For their computation, we use Lemma 4.5 and the j^{th} derivative of $\widetilde{H}_{n,p}(x)$ $(p = 1, 2)$ as defined by (18)-(19).

4.3 Polynomial and Factorial Polynomial Solutions for $\lambda_0 = 1$ of Multiplicity $m \ge 1$

Suppose that $\lambda_i \ne 1$ for all i $(0 \le i \le r-1)$. Then (14) and (21) imply, respectively, that the Asveld polynomials $P_j(n)$ $(0 \le j \le d)$ are of degree j and the Asveld factorial polynomials $\widetilde{P}_j(n)$ $(0 \le j \le d)$ are of degree j. Meanwhile, for $\lambda_0 = 1$, (15) and (22) show that $P_j(n)$ and $\widetilde{P}_j(n)$ $(0 \le j \le d)$ may be of degree $j+1$. More generally, an extension of Theorems 4.2 and 4.6 may be derived by the same method using, respectively,

$$G_n(x) = \sum_{i=0}^{m-1}\sum_{k=r}^{n}\alpha_{0,i}(n-k)^i e^{kx}$$

instead of $G_n(x)$ and

$$\widetilde{G}_n(x) = (-1)^j\sum_{i=0}^{m-1}\alpha_{0,i}\sum_{k=r}^{n}(n-k)^i x^{-k+j-1}$$

instead of $\widetilde{G}_n(x)$ of (19).

More precisely, we have the following result.

Theorem 4.7: Let $\{T_n\}_{n\ge 0}$ be a solution of (1) and suppose that $\lambda_0 = 1$ has multiplicity $m \ge 1$, and the other characteristic roots $\lambda_1, \ldots, \lambda_s$ (where $s = r - m - 1$) are simple.

(a) For $C(n) = n^j$, the elementary polynomial solution $\{P_j(n)\}_{n\ge 0}$ of (1) is given by

$$P_j(n) = \sum_{k=0}^{j+m}\nu_{j,k}n^k + \sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}g_2^{(j-i)}(0)\right\}n^k,$$

where $\nu_{j,k}$ are constants (real or complex numbers) and

$$g_2(x) = \sum_{i=1}^{s}\frac{\alpha_i}{e^x - \lambda_i}.$$

(b) For $C(n) = n^{(j)}$, the elementary factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n\ge 0}$ of (1) is given by

$$\widetilde{P}_j(n) = \sum_{k=0}^{j+m}\nu_{j,k}n^{(k)} + \sum_{k=0}^{j}\left\{\sum_{i=k}^{j}\binom{i}{j}\binom{k}{i}\widetilde{g}_2^{(j-i)}(1)\right\}n^{(k)},$$

where $\nu_{j,k}$ are constants (real or complex numbers) and

$$\widetilde{g}_2(x) = (-1)^j \sum_{i=1}^{s} \frac{\alpha_i}{1 - x\lambda_i}.$$

Theorem 4.7 shows that $P_j(n)$ and $\widetilde{P}_j(n)$ may be of degree $j+m$, where m is the multiplicity of $\lambda_0 = 1$.

4.4 Solutions of (1) for General $\{C_n\}_{n\geq 0}$

In the general situation, polynomial and factorial polynomial solutions of (1) are as follows.

Proposition 4.8: Let $\{T_n\}_{n\geq 0}$ be a solution of (1) and suppose that the characteristic roots $\lambda_0, \ldots, \lambda_{r-1}$ are simple. Then:

(a) For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the particular fundamental polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is given by $P(n) = \sum_{j=0}^{d} \beta_j S_n^{(j)}(0)$. More precisely, $P(n) = \sum_{j=0}^{d} \beta_j P_j(n)$, where $P_j(n)$ is given by (14) if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$ and (15) if $\lambda_0 = 1$.

(b) For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the particular fundamental factorial polynomial solution $\{\widetilde{P}(n)\}_{n\geq 0}$ of (1) is given by $\widetilde{P}(n) = \sum_{j=0}^{d} \beta_j \widetilde{S}_n^{(j)}(1)$. More precisely, $\widetilde{P}(n) = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n)$, where $\widetilde{P}_j(n)$ is given by (21) if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$ and by (22) if $\lambda_0 = 1$.

From Lemma 4.1 and Theorem 4.2, we derive that in the polynomial case the elementary fundamental particular solutions of (1) are

$$W_{j,n}^{\langle ps \rangle} = P_j(n) + \sum_{i=0}^{r-1} v_i^{(j)}(0) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$, where $P_j(n)$ is given by (14) and

$$v_i(x) = \frac{\alpha_i e^{rx}}{\lambda_i - e^x}$$

And if $\lambda_0 = 1$, we have

$$W_{j,n}^{\langle ps \rangle} = P_j(n) + \sum_{i=0}^{r-1} u_i^{(j)}(0) \lambda_i^{n-r+1},$$

where $P_j(n)$ is given by (15) above. For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the fundamental particular solution $\{W_n^{\langle ps \rangle}\}_{n\geq 0}$ is given by

$$W_n^{\langle ps \rangle} = \sum_{j=0}^{d} \beta_j W_{j,n}^{\langle ps \rangle}.$$

In the same manner, Lemma 4.4 and Theorem 4.6 imply that, for the factorial polynomial case, elementary fundamental particular solutions are

$$\widetilde{W}_{j,n}^{\langle ps \rangle} = \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \widetilde{v}_i^{(j)}(1) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$, where $\widetilde{P}_j(n)$ is given by (21) above, and

$$\widetilde{v}_i(x) = (-1)^j \frac{\alpha_i x^{j-r}}{\lambda_i x - 1}.$$

And if $\lambda_0 = 1$, we have

$$\widetilde{W}_{j,n}^{\langle ps\rangle} = \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \widetilde{v}_i^{(j)}(1)\lambda_i^{n-r+1},$$

where $\widetilde{P}_j(n)$ is given by (22) above. For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the fundamental particular solution $\{\widetilde{W}_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) may be expressed as

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{W}_{j,n}^{\langle ps\rangle}.$$

More precisely, Lemmas 4.1 and 4.4, Theorems 4.2 and 4.6, and Proposition 4.8 imply

Proposition 4.9: Let $\{T_n\}_{n\geq 0}$ be a solution of (1) and suppose that the characteristic roots $\lambda_0, \ldots, \lambda_{r-1}$ are simple. Then

(a) For $C(n) = \sum_{j=0}^{d} \beta_j n^j$, the fundamental particular solution $\{W_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is

$$W_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j P_j(n) + \sum_{i=0}^{r-1} \left(\sum_{j=0}^{d} \beta_j v_i^{(j)}(0) \right) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$, where

$$v_i(x) = \frac{\alpha_i e^{rx}}{e^x - \lambda_i}$$

and $P_j(n)$ is given by (14). And if $\lambda_0 = 1$, we have

$$W_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j P_j(n) + \sum_{i=1}^{r-1} \left(\sum_{j=0}^{d} \beta_j v_i^{(j)}(0) \right) \lambda_i^{n-r+1},$$

where $P_j(n)$ is given by (15).

(b) For $C(n) = \sum_{j=0}^{d} \beta_j n^{(j)}$, the fundamental particular solution $\{\widetilde{W}_n^{\langle ps\rangle}\}_{n\geq 0}$ of (1) is

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n) + \sum_{i=0}^{r-1} \left(\sum_{j=0}^{d} \beta_j \widetilde{v}_i^{(j)}(1) \right) \lambda_i^{n-r+1}$$

if $\lambda_i \neq 1$ for all i $(0 \leq i \leq r-1)$, where

$$\widetilde{v}_i(x) = (-1)^j \frac{\alpha_i x^{j-r}}{\lambda_i x - 1}$$

and $\widetilde{P}_j(n)$ is given by (21). And if $\lambda_0 = 1$, we have

$$\widetilde{W}_n^{\langle ps\rangle} = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n) + \sum_{i=1}^{r-1} \left(\sum_{j=0}^{d} \beta_j \widetilde{v}_i^{(j)}(1) \right) \lambda_i^{n-r+1},$$

where $\widetilde{P}_j(n)$ is given by (22).

5. CONCLUDING REMARKS

Remark 5.1: Relation with Genocchi and Bernoulli Numbers. In the j^{th} derivative of $H_{p,n}(x)$ $(p = 1, 2)$ given by (11)-(12) appears the k^{th} $(0 \leq k \leq j)$ derivative of functions $u_i(x) = \frac{\alpha_i}{e^x - \lambda_i}$. Let $u(x) = \frac{\alpha}{e^x - \lambda}$, where $\lambda < 0$, then

$$u(x) = \nu \frac{1}{e^{x+\beta}+1} = \frac{2\nu}{x+\beta} v(x+\beta),$$

where $\nu = -\frac{\alpha}{\lambda}$, $\beta = -\ln(-\lambda)$, and $v(t) = \frac{2t}{e^t+1}$. The Genocchi numbers G_n $(n \geq 0)$ are defined by

$$\sum_{n=0}^{+\infty} G_n \frac{t^n}{n!} = v(t)$$

(see [5] and [8]). So, because $G_0 = 0$, we have

$$u(x) = \frac{1}{2\nu} \sum_{n=0}^{+\infty} G_{n+1} \frac{(x+\beta)^n}{n!} = \frac{1}{2\nu} \sum_{n=0}^{+\infty} \left(\sum_{k=n}^{+\infty} \frac{G_{n+1}}{(n-k)!(k+1)} \beta^{k-n} \right) \frac{x^n}{n!}.$$

Particularly, for $\lambda = -1$, we have

$$u(x) = \frac{1}{2\alpha} \sum_{n=0}^{+\infty} G_{n+1} \frac{x^n}{n!}.$$

If $\lambda_0 = 1$ is a simple characteristic root, we may take, for any $x \neq 0$, $G_n(x) = \alpha_0 h_n(x) w(x)$, where $h_n(x) = \frac{e^{(n-r+1)x}-1}{x}$ and $w(x) = \frac{x}{e^x-1}$. Expansion series of these two functions are

$$h_n(x) = \sum_{k=0}^{+\infty} \frac{(n-r+1)^k}{k+1} \frac{x^k}{k!}, \quad w(x) = \sum_{k=0}^{+\infty} B_k \frac{x^k}{k!},$$

where B_k are the Bernoulli numbers (see, e.g., [5] and [8]). Then Leibnitz's formula

$$G_n^{(k)}(x) = \alpha_0 \sum_{i=0}^{k} \binom{i}{k} h_n^{(i)} u^{(k-i)}(x)$$

implies that

$$G_n^{(k)}(0) = \alpha_0 \sum_{i=0}^{k} \binom{i}{k} \frac{(n-r+1)^i}{i+1} B_{k-i}.$$

Hence, Asveld's polynomials $P_j(n)$ $(0 \leq j \leq d)$ depend on the Genocchi and Bernoulli numbers when $\lambda < 0$ or $\lambda_0 = 1$.

Remark 5.2: Degree of $P_j(n)$ and $\widetilde{P}_j(n)$. Theorems 4.2, 4.6, and 4.7 show that Asveld's polynomials $P_j(n)$ and factorial polynomials $\widetilde{P}_j(n)$ $(0 \leq j \leq d)$ are of degree $j+m$, where m is the multiplicity of $\lambda_0 = 1$. This property is established by the two last authors using an alternative method for solving (1), which is the subject of another paper.

***Remark 5.3: The Case of Multiplicities* ≥ 1.** In Section 4 we considered that the characteristic roots are simple except for Theorem 4.7, where $\lambda_0 = 1$ is supposed of multiplicity $m \geq 1$. The problem is to derive the particular polynomial or factorial polynomial solutions of (1) using the method of Section 3 when the characteristic roots $\lambda_0, \ldots, \lambda_p$ $(p \leq r-1)$ are of arbitrary multiplicities $m_0, \ldots, m_p$.

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