ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-935 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI***

Prove that

$$8\sin\left(\frac{F_3}{2}\right)\sin\left(\frac{F_9}{2}\right)\sin\left(\frac{F_{12}}{2}\right) < 1$$

where the arguments are measured in degrees.

B-936 ***Proposed by José Luis Diaz & Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let n be a nonnegative integer. Show that the equation

$$x^5 + F_{2n}x^4 + 2(F_{2n} - 2F_{n+1}^2)x^3 + 2F_{2n}(F_{2n} - 2F_{n+1}^2)x^2 + F_{2n}^2 x + F_{2n}^3 = 0$$

has only integer roots.

B-937 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove the following identities:

(a) $(F_n)^2 + (F_{n+1})^2 + 4(F_{n+2})^2 = (F_{n+3})^2 + (L_{n+1})^2$;

(b) $(L_n)^2 + (L_{n+1})^2 + 4(L_{n+2})^2 = (L_{n+3})^2 + (5F_{n+1})^2$.

B-938 ***Proposed by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC***

Find the smallest positive integer k for which the given series converges and find its sum:

(a) $\sum_{n=1}^{\infty} \frac{nF_n}{k^n}$;

(b) $\sum_{n=1}^{\infty} \frac{nL_n}{k^n}$.

B-939 ***Proposed by N. Gauthier, Royal Military College of Canada***

For $n \geq 0$ and s arbitrary integers, with

$$f(l, m, n) \equiv f(l, m) = (-1)^{n-l}\binom{n}{l}\binom{n}{m},$$

prove the following identities:

(a) $2^n F_{n+s} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m) F_{l+s}$;

(b) $3 \cdot 2^{n-1} n F_{n+s+2} = \sum_{l=0}^{4n} \sum_{m=0}^{\lfloor l/3 \rfloor} f(l-3m, m)[(l-2m)F_{l+s} + mF_{l+s-1}]$.

SOLUTIONS

A Relatively Prime Fibonacci Couple

B-921 ***Proposed by the editors***
(Vol. 39, no. 3, June-July 2001)

Determine whether or not $F_{6n} - 1$ and $F_{6n-3} + 1$ are relatively prime for all $n \geq 1$.

Solution by Russell Jay Hendel, Towson University, Baltimore, MD

We go beyond the problem requirements by also providing explicit formulas for the relative primeness.

Recall that two integers a and b are relatively prime if and only if there exist integers x and y such that

$$ax + by = 1. \tag{1}$$

Accordingly, let

$$a = F_{6n} - 1, \quad a = F_{6n} - 1,$$
$$b = F_{6n-3} + 1. \, b = F_{6n-3} + 1.$$

The parallel processor algorithm of Hendel [2] motivates defining

$$x = F_{6n-5} - \{F_{6n-4} - F_{6n-10} - 4\} / 16,$$
$$y = \{F_{6n+3} + F_{6n+1} - F_{6n-3} - F_{6n-5} - 12\} / 16.$$

Using periodicity properties of the Fibonacci sequence modulo 16, it is straightforward to verify that x and y are in fact integers.

Using these definitions of x and y, (1) can be proven for all n by using the Verification Theorem of Dresel [1]. We need only check (1) for the first values of n and this is easily done by hand calculator. For example, when $n = 3$, (1) yields the explicit identity $2583*211-611*892=1$.

References

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5:**169-84. Ed. G. Bergum, et al. Dordrecht: Kluwer, 1993.
2. R. J. Hendel. "A Fibonacci Problem Classification Scheme Useful to Undergraduate Pedagogy." In *Applications of Fibonacci Numbers* **5:**289-304. Dordrecht: Kluwer, 1993.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Lake Superior State University Problem Group, H.-J. Seiffert, Gabriela & Pantelimon Stănică (jointly), and the proposers.

A Prime Search

B-922 ***Proposed by Irving Kaplansky, Math. Sciences Research Institute, Berkeley, CA (Vol. 39, no. 3, June-July 2002)***

Determine all primes p such that the Fibonacci numbers modulo p yield all residues.

Solution by Pantelimon Stănică, Auburn University, Montgomery, AL

In *The Fibonacci Quarterly* **6.2** (1968):139-41 ("Fibonacci Sequence Modulo m"), A. P. Shah proved that if p is a prime and $p \equiv 1, 9 \pmod{10}$ then the Fibonacci sequence does not form a complete residue modulo p.

In *The Fibonacci Quarterly* **8.3** (1970):000-00 ["Fibonacci Sequence Modulo a Prime $p \equiv 3 \pmod 4$"], G. Bruckner proved the same for the remaining cases if $p > 7$. Therefore, the Fibonacci sequence modulo p yields all residues if and only if $p = 2, 3, 5, 7$ by an easy calculation and using the above references.

In *The Fibonacci Quarterly* **38.3** (2000):272-81 ("Complete and Reduced Residue Systems of Second-Order Recurrences Modulo p"), H.-C. Li proved that even the generalized Fibonacci sequence with parameters $(a, 1)$ does not form a complete residue system modulo $p > 5$.

L. A. G. Dresel also referred to the G. Bruckner reference.

Also solved by P. Bruckman, L. A. G. Dresel, and the proposer.

The Fraction Continues

B-923 ***Proposed by José Luis Diaz & Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 3, June-July 2002)***

Let α_l be the l^{th} convergent of the continued fractional expansion:

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Prove that

(a) $\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{l+k} \geq [F_n\alpha_l + F_{n-1}]^{1/n}$,

(b) $\alpha_k^l = \sum_{j=0}^{k}\binom{k}{j}\frac{1}{\alpha_{l-1}^j}$ for all $k \in \mathbb{N}$.

Solution by Paul S. Bruckman, Sacramento, CA

The readers of this journal will readily recognize the following result:

$$\alpha_j = F_{j+1} / F_j$$

(for typographical clarity, the notation is modified).

(a) Let

$$A(n, j) = 1/n\sum_{k=0}^{n-1}\alpha_{j+k}.$$

Note that $A(n, j)$ is the arithmetic average (A.M.) of the quantities $\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+n-1}$. By the A.M.-G.M. inequality,

$$A(n, j) \geq G(n, j) \equiv \left(\prod_{k=0}^{n-1}\alpha_{j+k}\right)^{1/n}.$$

Note that $G(n, j) = (F_{j+n} / F_j)^{1/n}$.

Also,

$$F_n\alpha_j + F_{n-1} = (F_nF_{j+1} + F_{n-1}F_j) / F_j = F_{j+n} / F_j.$$

Thus, $A(n, j) \geq (F_n\alpha_j + F_{n-1})^{1/n}$. Q.E.D.

(b) Let

$$S(k, j) = \sum_{i=0}^{k} {}_kC_i(\alpha_{j-1})^{-i}.$$

Then

$$S(k, j) = (1 + 1/\alpha_{j-1})^k = (1 + F_{j-1} / F_j)^k$$
$$= \{(F_j + F_{j-1}) / F_j\}^k = (F_{j+1} / F_j)^k = (\alpha_j)^k.$$

This corrects the statement of this part of the problem.

Also solved by H.-J. Seiffert (essentially the same as the featured solution) and the proposer.

A Generalization of a Lucas Numbers Identity

A B-924 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 3, June-July 2001)***

For n an arbitrary integer, the following identity is easily established for Lucas numbers:

$$L_{2n+2} + L_{2n-2} = 3L_{2n}. \tag{1}$$

Consider the Fibonacci and Lucas polynomials, $\{F_n(u)\}_{n=0}^{\infty}$ and $\{L_n(u)\}_{n=0}^{\infty}$, defined by

$$F_0(u) = 0,\ F_1(u) = 1,\ F_{n+2}(u) = uF_{n+1}(u) + F_n(u),$$

and

$$L_0(u) = 2,\ L_1(u) = u,\ L_{n+2}(u) = uL_{n+1}(u) + L_n(u),$$

respectively. The corresponding generalization of (1) is

$$L_{2n+1}(u) + L_{2n-2}(u) = (u^2+2)L_{2n}(u). \tag{2}$$

For m a nonnegative integer, with the convention that a discrete sum with a negative upper limit is identically zero, prove the following generalization of (2):

$$\begin{aligned}&(n+1)^{2m}L_{2n+2}(u) + (n-1)^{2m}L_{2n-2}(u)\\ &= (u^2+2)\left[\sum_{l=0}^{m}\binom{2m}{2l}n^{2l}\right]L_{2n}(u) + u(u^2+4)\left[n\sum_{l=0}^{m-1}\binom{2m}{2l+1}n^{2l}\right]F_{2n}(u).\end{aligned} \tag{3}$$

Also prove the following companion identity:

$$\begin{aligned}&(n+1)^{2m+1}F_{2n+2}(u) + (n-1)^{2m+1}F_{2n-2}(u)\\ &= u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}n^{2l}\right]L_{2n}(u) + (u^2+2)\left[n\sum_{l=0}^{m-1}\binom{2m+1}{2l+1}n^{2l+1}\right]F_{2n}(u).\end{aligned} \tag{4}$$

Solution by H.-J. Seiffert, Berlin, Germany

In (4), the upper index in the second sum on the right-hand side must be replaced by m.

It is known [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, equations (3.23), (2.2), (2.1), and (3.22)] that

$$(u^2+2)L_{2n}(u) = L_{2n+2}(u) + L_{2n-2}(u), \tag{5}$$

$$(u^2+4)F_{2n}(u) = L_{2n-1}(u) + L_{2n+1}(u), \tag{6}$$

$$L_{2n}(u) = F_{2n-1}(u) + F_{2n+1}(u), \tag{7}$$

$$(u^2+2)F_{2n}(u) = F_{2n+2}(u) + F_{2n-2}(u); \tag{8}$$

note that (5) is the corrected version of (2).

$$\sum_{l=0}^{m}\binom{2m}{2l}n^{2l} = \frac{(n+1)^{2m}+(n-1)^{2m}}{2}, \tag{9}$$

$$\sum_{l=0}^{m-1}\binom{2m}{2l+1}n^{2l+1} = \frac{(n+1)^{2m}-(n-1)^{2m}}{2}, \tag{10}$$

$$\sum_{l=0}^{m}\binom{2m+1}{2l}n^{2l} = \frac{(n+1)^{2m+1}-(n-1)^{2m+1}}{2}, \tag{11}$$

$$\sum_{l=0}^{m}\binom{2m+1}{2l+1}n^{2l+1} = \frac{(n+1)^{2m+1}+(n-1)^{2m+1}}{2}. \tag{12}$$

Proof of (3): In view of (5), (6), (9), and (10), we must show that

$$\begin{aligned}&(n+1)^{2m}L_{2n+2}(u) + (n-1)^{2m}L_{2n-2}(u)\\ &= \frac{(n+1)^{2m}+(n-1)^{2m}}{2}(L_{2n+2}(u) + L_{2n-2}(n)) + \frac{(n+1)^{2m}-(n-1)^{2m}}{2}(uL_{2n-1}(u) + uL_{2n+1}(u)),\end{aligned}$$

which is true because

$$L_{2n-2}(u)+uL_{2n-1}(u)+uL_{2n+1}(u)=L_{2n+2}(u)$$

and, equivalently,

$$L_{2n+2}(u)-uL_{2n-1}(u)-uL_{2n+1}(u)=L_{2n-2}(u).$$

Proof of (4): This is easily verified by applying (7), (8), (11), and (12), and using

$$uF_{2n-1}(u)+uF_{2n+1}(u)+F_{2n-2}(u)=F_{2n+2}(u).$$

Also solved by P. Bruckman and the proposer.

The Gandhi Polynomials

In response to Paul Bruckman's question, Reiner Martin sent the following remark:

In the August 2001 issue of *The Fibonacci Quarterly*, Paul Bruckman asks whether the polynomials $P(r,n)$ given by $P(1,n)=n$ and $P(r+1,n)=n^2(P(r,n)-P(r,n-1))$ are new to the literature.

Indeed, these polynomials (or, rather, a trivial variation thereof) are known as Gandhi polynomials. References are:

[1] D. Dumont, "Sur une conjecture de Gandhi concernant les nombres de Genocchi," *Discrete Mathematics* **1** (1972):321-27.

[2] D. Dumont, "Interpretations combinatoires des nombres de Genocchi," *Duke Math. Journal* **41** (1974):305-18.

Identifying these polynomials illustrates the usefulness of Sloane's On-Line Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences/). Entering the first few nonzero coefficients as $1,-1,2,3,-8,6,-17,54,-60,24$ into the database yields a hit (up to signs) with the sequence A036970 (triangle of coefficients of Gandhi polynomials), where the references can be found.

We wish to belatedly acknowledge the solution to problem B-915 by Walther Janous. In fact, his solution gives a sharper inequality that will appear in a separate proposal.

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