ON THE REPRESENTATION OF THE INTEGERS AS A DIFFERENCE OF SQUARES

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1. INTRODUCTION

In recent times a number of authors (see [1]-[4]) have revisited the well-known results of Fermat and Jacobi in connection with the polygonal representation of the integers. In the papers cited an alternate derivation for such formulas giving the total number of representations of an integer as the sum of either two triangular or square numbers was provided. These enumerations, which are given in terms of elementary divisor functions, were deduced as a consequence of the Gauss-Jacobi triple product identity. In contrast to sums of polygonal numbers, the author has investigated within [5] the representation of the integers as a difference of two triangular numbers. By use of a purely combinatorial argument, it was shown that the number of such representations of an integer n was exactly equal to the number of odd divisors of n. In this note we propose to extend the methods employed in [5] to the case of squares to prove the following result.

Theorem 1.1: The number $s(n)$ of representations of a positive integer as a difference of the squares of two nonnegative integers is given by

$$s(n)=\frac{1}{2}\left(d_0(n)+(-1)^{n+1}d_1(n)+\frac{1+(-1)^{d(n)+1}}{2}\right), \tag{1}$$

where $d(n)$ is the total number of divisors of n and, for each $i\in\{0,1\}$,

$$d_i(n)=\sum_{d|n,\, d\equiv i \bmod 2} 1.$$

To facilitate the result, we shall need a preliminary definition and technical lemma.

Definition 1.1: For a given $n\in\mathbb{N}\setminus\{0\}$, a factorization $n=ab$, with $a,b\in\mathbb{N}\setminus\{0\}$ is said to be nontrivial if $a\neq 1,n$. Two such factorizations, $a_1b_1=a_2b_2=n$, are distinct if $a_1\neq a_2, b_2$.

The following result, which concerns counting the total number of distinct nontrivial factorizations, $ab=n$, may be known; however, interested readers can consult [5] for a proof.

Lemma 1.1: Let n be an integer greater than unity and $d(n)$ the number of divisors of n. Then the total number $N(n)$ of nontrivial distinct factorizations of n is given by

$$N(n)=\begin{cases}\dfrac{d(n)-2}{2} & \text{for nonsquare } n,\\[2ex] \dfrac{d(n)-1}{2} & \text{for square } n.\end{cases}$$

2. PROOF OF THEOREM 1.1

Our first goal will be to determine whether, for a given $n \in \mathbb{N} \setminus \{0\}$, there exists $x, y \in \mathbb{N}$ such that $n = x^2 - y^2$. To analyze the solvability of this diophantine equation, suppose $n = ab$, where $a, b \in \mathbb{N} \setminus \{0\}$, and consider the following system of simultaneous linear equations

$$\begin{aligned} x - y &= a \\ x + y &= b \end{aligned} \tag{2}$$

whose general solution is given by

$$(x, y) = \left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Now, for there to exist a representation of n as a difference of two squares, one must be able to find a factorization $ab = n$ for which (2) will yield a solution (x, y) in integers.

Remark 2.1: We note that it is sufficient to consider only (2) since, if for a chosen factorization $ab = n$ an integer solution pair (x, y) is found, then the corresponding representation $n = x^2 - y^2$ is also obtained if the right-hand side of (2) is interchanged. Indeed, one finds upon solving

$$\begin{aligned} x' - y' &= b \\ x' + y' &= a \end{aligned}$$

that $x' = \frac{a+b}{2}$ and $y' = \frac{a-b}{2}$. Thus, $x' = x$ while $y' = -y$, which yields an identical difference of squares representation.

We deal with the existence or otherwise of those factorizations $ab = n$, which gives rise to the integer solution pair (x, y) of (2). It is clear from the general solution of (2) that, for x to be a positive integer a, b must at least be chosen so that $a + b$ is an even integer. Clearly, this can only be achieved if a and b are of the same parity. Furthermore, such a chose of a and b will also ensure that $y = x - a$ is also an integer. With this reasoning in mind, it will be convenient to consider the following cases separately.

Case 1: $n = 4k + 2$, $k \in \mathbb{N}$. In this instance, if $ab = 2(2k+1)$, then one cannot possibly find an a and b of the same parity, so no integer solution (x, y) of (2) can be found. Consequently, $s(4k+2) = 0$.

Case 2: $n \neq 4k + 2$. Clearly, $n = 2^m(2k+1)$ for some $n \in \mathbb{N} \setminus \{1\}$ and $s \in \mathbb{N}$. Considering first $m = 0$, it is immediate that all factorizations $ab = 2k + 1$ will produce integer solutions to (2) since a and b are odd. Alternatively, when $m > 1$ one can always construct, for every factorization $cd = 2k + 1$, an a and b of the form $(a, b) = (2^i c, 2^{m-i} d)$ with $i \in \{1, 2, \ldots., m-1\}$ that will produce an integer solution of (2). Hence, for the m and k prescribed above, one can conclude that $s(2^m(2k+1)) > 0$.

Having determined the set of integers n which are of the form $n = x^2 - y^2$, we can now address the problem of finding the exact number $s(n)$ of such representations. Primarily, this will entail determining whether any duplication occurs between the representations generated from the various distinct factorizations discussed in Case 2. To this end, we need to demonstrate that if in $\mathbb{Z} \setminus \{0\}$ $a_i b_i = a_j b_j$, with $a_i \neq a_j, b_j$ for $i \neq j$, then one has $a_i + b_i \neq a_j + b_j$. Suppose to the contary that $a_i + b_i = a_j + b_j$, then there must exist an $r \in \mathbb{Z} \setminus \{0\}$ such that $a_j = a_i + r$ and $b_i = b_j + r$.

Substituting these equations into the equality $a_i b_i = a_j b_j$, one finds $a_i(b_j + r) = (a_i + r)b_j$. Hence, r must be a nonzero integer solution of

$$r(a_i - b_j) = 0. \tag{3}$$

However, this is impossible as $r = 0$ is the only possible solution of (3) since $a_i - b_j \neq 0$; a contradiction. Consequently, if for two distinct factorizations $a_i b_i = a_j b_j = n$, one solves (2) to produce corresponding integer solutions (x_i, y_i) and (x_j, y_j), then we must have $x_i = (a_i + b_i)/2 \neq (a_j + b_j)/2 = x_j$ and so $y_i = x_i - a \neq x_j - a = y_j$. Thus, in order to calculate $s(n)$ for an $n \neq 4k+2$, one must determine the total number of distinct factorizations discussed in Case 2. Considering when n is odd we have, from Lemma 1.1, $N(n)$ nontrivial distinct factorizations. However, as $(a,b) = (1,n)$ contributes a representation, one has $s(n) = N(n)+1$, which we write here using $d(n) = d_0(n) + d_1(n)$ as

$$s(n) = \begin{cases} \frac{1}{2}(d_0(n) + d_1(n)) & \text{for nonsquare } n, \\ \frac{1}{2}(d_0(n) + d_1(n) + 1) & \text{for square } n. \end{cases} \tag{4}$$

Suppose n is even, then, as was observed previously, $s(n)$ is equal to the total number of distinct factorizations $a_i b_i = n$ were both a_i and b_i are even. Denoting the number of distinct factorizations of $n = a_i b_i$ with a_i and b_i of opposite parity by $N'(n)$, observe that $s(n)$ must be equal to the difference between the number of distinct factorizations of n and $N'(n)$, that is, $s(n) = N(n) + 1 - N'(n)$. To determine $N'(n)$, consider an arbitrary factorization (c_i, d_i) **(possibly trivial)** of the odd number $2^{-m}n$. Now, if $2^{-m}n$ is not a perfect square, then $(2^m c_i, d_i)$ and $(c_i, 2^m d_i)$ must be distinct factorizations of n, which cannot be duplicated by the use of an alternate factorization (c_j, d_j) of $2^{-m}n$. Thus, from Lemma 1.1, there are $2\left(\frac{d(2^{-m}n)}{2}\right)$ distinct factorizations $a_i b_i = n$ having a_i and b_i of opposite parity. Similarly, if $2^{-m}n$ is a square, then $(2^m c_i, d_i)$ and $(c_i, 2^m d_i)$ will be distinct factorizations provided $c_i \neq d_i$, and so again by Lemma 1.1 we have, counting the single contribution from $(2^m c_i, c_i)$, precisely $2\left(\frac{d(2^{-m}n)+1}{2} - 1\right) + 1$ distinct factorizations $a_i b_i = n$ with a_i and b_i of opposite parity. Consequently, in any case, $N'(n) = d(2^{-m}n)$. Now, observing that $d(2^{-m}n) = d_1(n)$ and $d(n) = d_0(n) + d_1(n)$, we obtain, for an even $n \neq 4k+2$, the following expression:

$$s(n) = \begin{cases} \frac{1}{2}(d_0(n) - d_1(n)) & \text{for nonsquare } n, \\ \frac{1}{2}(d_0(n) - d_1(n) + 1) & \text{for square } n. \end{cases} \tag{5}$$

Recalling that $d(n)$ is odd if and only if n is a square, we find

$$\frac{1 + (-1)^{d(n)+1}}{2} = \begin{cases} 0 & \text{for nonsquare } n, \\ 1 & \text{for square } n. \end{cases}$$

Thus, one can combine equations (4) and (5) into a single expression independent of the parity of n as indicated in (1). Finally, we show that (1) holds for $n = 4k+2$. In this instance, as n cannot be a square, $\frac{1}{2}(1 + (-1)^{d(n)+1}) = 0$; moreover, $d_0(2(2k+1)) = d_1(2(2k+1))$, since every odd divisor d of n is in one-to-one correspondence with an even divisor of n, namely, $2d$. Thus, from (1), we find that $s(2(2k+1)) = \frac{1}{2}(d_0(2(2k+1)) - d_1(2(2k+1))) = 0$ as required.

Example 2.1: For a given integer n, whose prime factorization is known, one can determine all of the $s(n)$ representations of n as a difference of squares from the factorizations $ab = n$ with $a \geq b > 0$ and $a \equiv \pm b \pmod 2$, using $(x, y) = (\frac{a+b}{2}, \frac{a-b}{2}$. To illustrate this, we shall calculate the representations in the case of a square and nonsquare number. Beginning with, say $n = 2^2 \cdot 5 \cdot 7$, we have that $d_0(140) = 8$, $d_1(140) = 4$, and $d(140)$ even, so $s(140) = \frac{1}{2}(8-4) = 2$. Thus, from the two factorizations $(a, b) \in \{(2 \cdot 7, 2 \cdot 5), (2 \cdot 5 \cdot 7, 2)\}$, we find that $140 = 12^2 - 2^2$, $36^2 - 34^2$. In the case of $n = (2 \cdot 5 \cdot 7)^2$, we have $d_0(4900) = 18$, $d_1(4900) = 9$, $d(4900)$ odd; therefore, $s(4900) = \frac{1}{2}(18 - 9 + 1) =$. Thus, again from the five factorizations $(a, b) \in \{(2 \cdot 5^2 \cdot 7^2, 2), (2 \cdot 5^2 \cdot 7, 2 \cdot 7), (2 \cdot 5 \cdot 7, 2 \cdot 5 \cdot 7), (2 \cdot 7^2, 2 \cdot 5^2), (2 \cdot 5 \cdot 7^2, 2 \cdot 5)\}$, we now obtain that $4900 = 1226^2 - 1224^2$, $250^2 - 240^2, 70^2 - 0^2, 74^2 - 24^2, 182^2 - 168^2$.

To conclude, we present a simple application of Theorem 1.1 for counting the number of those partitions of an integer whose summands form a sequence of consecutive odd integers. Note that for an odd integer n we do not count $n = 0 + n$ as a partition of the required type as zero is not an odd integer.

Corollary 2.1: If $p_o(n)$ denotes the number of partitions of a positive integer n having summands consisting of consecutive odd integers, then

$$p_o(n) = s(n) + \frac{(-1)^n - 1}{2}.$$

Proof: Recalling that the m^{th} perfect square is equal to the sum of the first m odd integers, one sees that the representation $n = x^2 - y^2$ gives a partition of the required form, provided that $x - y > 1$. Moreover, as a consecutive square difference representation can only occur for an odd integer, we clearly must have $p_o(n) = s(n) - 1$ for odd n and $p_o(n) = s(n)$ for even n. □

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