SEQUENCES RELATED TO RIORDAN ARRAYS

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1. INTRODUCTION

The concept of a Riordan array was defined in [4] as follows: Let $\mathscr{F} = \mathbf{R}[x]$ be a ring of formal power series with real coefficients in some indeterminate x. Let $g(x) \in \mathscr{F}$ and let $f(x) = \sum_{k=0}^{\infty} f_k x^k \in \mathscr{F}$ with $f_0 = 0$ (in this paper we assume $f_1 \neq 0$). Let $d_0(x) = g(x)$, $d_k = g(x)(f(x))^k$, and $d_{n,k} = [x^n]d_k(x)$, where $[x^n]d_k(x)$ means the coefficients of x^n in the expansion of $d_k(x)$ in x. Then an infinite lower triangular array, $D = \{d_{n,k} \mid k, n \in \mathrm{N}, k \leq n\}$, is obtained. We also write $D = (g(x), f(x))$ and call D a Riordan array. In this paper we obtain some new relations between two sequences and some new inverse relations by using Riordan arrays. Some results are a generalization of [2] and [3].

2. SEQUENCES RELATED TO RIORDAN ARRAYS

Let $a(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{F}$ and $D = (g(x), f(x))$. Let

$$h(x) = \frac{1}{g(x)} = \sum_{k=0}^{\infty} h_k x^k \in \mathscr{F},$$

$$A(x) = a(f(x)) = \sum_{k=0}^{\infty} A_k x^k \in \mathscr{F},$$

and

$$s(x) = g(x)A(x) = \sum_{k=0}^{\infty} s_k x^k \in \mathscr{F}.$$

Theorem 1: We have

$$A_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} d_{i,k} h_{n-i} \right) a_k. \tag{1}$$

Proof: By Theorem 1.1 in [5], we have

$$\sum_{k=0}^{\infty} d_{n,k} a_k = [x^n] g(x) a(f(x)) = s_n.$$

From $s(x) = g(x)A(x)$, $A(x) = s(x)h(x)$, we have

$$A_n = \sum_{i=0}^{n} s_i h_{n-i} = \sum_{i=0}^{n} \left(\sum_{k=0}^{\infty} d_{i,k} a_k \right) h_{n-i} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} d_{i,k} h_{n-i}) \right) a_k..$$

This completes the proof. □

Theorem 2: We have

$$a_n = \sum_{k=0}^{\infty} \bar{d}_{n,k} A_k, \tag{2}$$

where $\bar{d}_{n,k}$ can be obtained by using one of the following Lagrange inversion formulas (see [1], pp. 148-52):

$$\bar{d}_{n,k} = \frac{k}{n}[x^{n-k}]\left(\frac{f(x)}{x}\right)^{-n}; \tag{3}$$

$$\bar{d}_{n,k} = k\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{n+j}\binom{n-k}{j}f_1^{-n-j}[x^{n-k+j}](f(x))^j. \tag{4}$$

Proof: By $A(x) = a(f(x))$, we have $a(x) = A(\bar{f}(x))$, where $\bar{f}(f(x)) = f(\bar{f}(x)) = x$ and $\bar{f}(0) = 0$. By [1] and Theorem 1.1 in [5], we obtain $a_n = \sum_{k=0}^{\infty} \bar{d}_{n,k} A_k$, in which

$$\bar{d}_{n,k} = [x^n](\bar{f}(x))^k = \frac{k}{n}[x^{n-k}]\left(\frac{f(x)}{x}\right)^{-n}$$

or

$$\bar{d}_{n,k} = [x^n](\bar{f}(x))^k = k\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{n+j}\binom{n-k}{j}f_1^{-n-j}[x^{n-k+j}](f(x))^j.$$

This completes the proof. □

We can combine Theorems 1 and 2 to obtain a generator of an inverse relation.

Theorem 3: We have the following inverse relation,

$$\begin{cases} A_n = \displaystyle\sum_{k=0}^{\infty}\left(\sum_{i=0}^{n} d_{i,k}h_{n-i}\right)a_k, \\ a_n = \displaystyle\sum_{k=0}^{\infty} \bar{d}_{n,k}A_k, \end{cases} \tag{5}$$

where $\bar{d}_{n,k}$ can be obtained by using (3) or (4).

In addition, we obtain many new identities by using (1) or (2). The interested reader can consult [2] and [3].

Example 1: Let $g(x) = \frac{1}{1-ax}$ and $f(x) = \frac{b^l x^l}{(1-ax)^s}$. Then $h(x) = 1 - ax$ and

$$d_{n,k} = [x^n]\frac{1}{1-ax}\left(\frac{b^l x^l}{(1-ax)^s}\right)^k = b^{kl}a^{n-lk}\binom{n+(s-l)k}{sk}.$$

By (1), we have

$$A_n = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n} b^{kl}a^{i-lk}\binom{i+(s-l)k}{sk}h_{n-i}\right)a_k$$

$$= \sum_{k=0}^{\infty}\left(b^{kl}a^{n-lk}\binom{n+(s-l)k}{sk} - b^{kl}a^{n-lk}\binom{n+(s-l)k-1}{sk}\right)a_k = \sum_{k=0}^{\infty} b^{kl}a^{n-kl}\binom{n+(s-l)k-1}{sk-1}a_k.$$

By (2) and (3), we have

$$\bar{d}_{n,k} = \frac{k}{n}[x^{n-k}]\left(\frac{b^l x^l}{x(1-ax)^s}\right)^{-n} = (-1)^{nl-k}a^{nl-k}b^{-nl}\frac{k}{n}\binom{sn}{nl-k},$$

$$a_n = \sum_{k=0}^{\infty}(-a)^{nl-k}b^{-nl}\frac{k}{n}\binom{sn}{nl-k}A_k.$$

So we obtain the following inverse relation:

$$\begin{cases} A_n = \sum_{k=0}^{\infty} b^{kl}a^{n-kl}\binom{n+(s-l)k-1}{sk-1}a_k, \\ a_n = \sum_{k=0}^{\infty}(-a)^{nl-k}b^{-nl}\frac{k}{n}\binom{sn}{nl-k}A_k. \end{cases}$$

Letting $s=l=1$, $a=t$, and $b=s$, we can obtain Theorems 3 and 4 in [3]. □

Example 2: Let $D_1 = (1, \log(1-x)) = (d^1_{n,k})$ and $D_2 = (1, (1-e^x)) = (d^2_{n,k})$. Then

$$d^1_{n,k} = [x^n](\log(1-x))^k = (-1)^k[x^n]\left(\log\left(\frac{1}{1-x}\right)\right)^k = (-1)^k\frac{k!}{n!}s_1(n,k)$$

and

$$d^2_{n,k} = [x^n](1-e^x)^k = (-1)^k[x^n](e^x-1)^k = (-1)^k\frac{k!}{n!}s_2(n,k).$$

From $A(x) = a(\log(1-x))$, we find $a(x) = A(1-e^x)$. So by (1) we have

$$\begin{cases} A_n = \sum_{k=0}^{\infty}(-1)^k\frac{k!}{n!}s_1(n,k)a_k, \\ a_n = \sum_{k=0}^{\infty}(-1)^k\frac{k!}{n!}s_2(n,k)A_k, \end{cases}$$

where $s_1(n,k)$ and $s_2(n,k)$ are the Stirling numbers of both kinds and have the following generating functions (see [5]), respectively:

$$\left(\log\frac{1}{1-x}\right)^m = \sum_{n=0}^{\infty}\frac{m!}{n!}s_1(n,m)x^n; \quad (e^x-1)^m = \sum_{n=0}^{\infty}\frac{m!}{n!}s_2(n,m)x^n. \quad \square$$

3. SEQUENCES RELATED TO EXPONENTIAL RIORDAN ARRAYS

Let

$$f(x) = \sum_{k=0}^{\infty} f_k\frac{x^k}{k!}.$$

We introduce a new notation, $\langle x^k\rangle f(x) = f_k$, and assume $f_0 = 0$, $f_1 \neq 0$. Let

$$g(x) = \sum_{k=0}^{\infty} g_k\frac{x^k}{k!}, \; g_0 \neq 0.$$

For an infinite lower triangular array $E = \{e_{n,k} \,|\, n, k \in \mathrm{N},\ k \le n\}$, if

$$\mathrm{e}_{n,k} = \langle x^n \rangle g(x) \frac{(f(x))^k}{k!} \quad (k \geq 0),$$

for fixed k, then we write $E = \langle g(x), f(x) \rangle$ and say that $\langle g(x), f(x) \rangle$ is an exponential Riordan array.

Let

$$b(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$$

and let $E = \langle g(x), f(x) \rangle$ be an exponential Riordan array. Let

$$p(x) = \frac{1}{g(x)} = \sum_{k=0}^{\infty} p_k \frac{x^k}{k!}, \quad B(x) = b(f(x)) = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

and

$$q(x) = g(x)B(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}.$$

For the exponential Riordan arrays, we have the following theorem as Theorem 1.1 in [5].

Theorem 4: We have

$$\sum_{k=0}^{\infty} \mathrm{e}_{n,k} b_k = \langle x^n \rangle g(x) b(f(x)). \tag{6}$$

Proof:

$$\sum_{k=0}^{\infty} \mathrm{e}_{n,k} b_k = \sum_{k=0}^{\infty} \langle x^n \rangle g(x) \frac{(f(x))^k}{k!} b_k = \langle x^n \rangle g(x) b(f(x)). \quad \square$$

Example 3: Let $E = \langle \mathrm{e}^x, -x \rangle$ be an exponential Riordan array. Then

$$\mathrm{e}_{n,k} = \langle x^n \rangle \mathrm{e}^x \frac{(-x)^k}{k!} = (-1)^k \binom{n}{k}.$$

For

$$b(x) = \frac{\mathrm{e}^{ax} - \mathrm{e}^{bx}}{a-b} = \sum_{n=0}^{\infty} F_n \frac{x^n}{n!},$$

where $a, b = (1 \pm \sqrt{5})/2$ and F_n is the n^{th} Fibonacci number defined by $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$ (see [2]), by (6) we have

$$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} F_k = \langle x^n \rangle \mathrm{e}^x b(-x) = \langle x^n \rangle \mathrm{e}^x \frac{\mathrm{e}^{-ax} - \mathrm{e}^{-bx}}{a-b} = \langle x^n \rangle - b(x) = -F_n,$$

that is,

$$\sum_{k=0}^{\infty} (-1)^{k+1} \binom{n}{k} F_k = F_n.$$

This is (8) in [2]. $\square$

By (6), we can obtain many new identities. The interested reader can refer to the related documents.

Theorem 5: We have

$$B_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \mathrm{e}_{i,k} p_{n-i} \right) b_k. \tag{7}$$

Proof: The proof is similar to that of Theorem 1. □

Theorem 6: We have

$$b_n = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k, \tag{8}$$

where $\bar{e}_{n,k}$ can be obtained by using one of the following Lagrange inversion formulas (see [1], 148-52):

$$\bar{e}_{n,k} = \binom{n-1}{k-1} \langle x^{n-k} \rangle \left(\frac{f(x)}{x} \right)^{-n}. \tag{9}$$

$$\bar{e}_{n,k} = \frac{n!}{(k-1)!} \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{(n+j)(n-k+j)!} \binom{n-k}{j} f_1^{-n-j} \langle x^{n-k+j} \rangle (f(x))^j. \tag{10}$$

Proof: From $B(x) = b(f(x))$, we have $b(x) = B(\bar{f}(x))$, where $\bar{f}(f(x)) = f(\bar{f}(x)) = x$. So

$$b_n = \langle x^n \rangle B(\bar{f}(x)) = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k,$$

where

$$\begin{aligned} \bar{e}_{n,k} &= \langle x^n \rangle \frac{(\bar{f}(x))^k}{k!} = [x^n] \frac{n!}{k!} (\bar{f}(x))^k \\ &= \frac{(n-1)!}{(k-1)!} [x^{n-k}] \left(\frac{f(x)}{x} \right)^{-n} = \binom{n-1}{k-1} \langle x^{n-k} \rangle \left(\frac{f(x)}{x} \right)^{-n} \end{aligned}$$

or

$$\begin{aligned} \bar{e}_{n,k} &= [x^n] \frac{n!}{k!} (\bar{f}(x))^k = \frac{n!}{k!} k \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{n+j} \binom{n-k}{j} f_1^{-n-j} [x^{n-k+j}] (f(x))^j \\ &= \frac{n!}{(k-1)!} \binom{2n-k}{n} \sum_{j=1}^{n-k} \frac{(-1)^j}{(n+j)(n-k+j)!} \binom{n-k}{j} f_1^{-n-j} \langle x^{n-k+j} \rangle (f(x))^j. \quad \square \end{aligned}$$

Theorem 7: As in Theorem 3, we have the following inverse relation,

$$\begin{cases} B_n = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} e_{i,k} p_{n-i} \right) b_k, \\ b_n = \sum_{k=0}^{\infty} \bar{e}_{n,k} B_k, \end{cases}$$

where $\bar{e}_{n,k}$ can be obtained by using (9) or (10).

Example 4: Let $\langle g(x), f(x) \rangle = \left\langle 1, \log \frac{1}{1-x} \right\rangle$. Then

$$e_{n,k} = \langle x^n \rangle \frac{(\log \frac{1}{1-x})^k}{k!} = \frac{1}{k!} \langle x^n \rangle \left(\log \frac{1}{1-x} \right)^k = s_1(n,k).$$

By (7), we have

$$B_n = \sum_{k=0}^{\infty} s_1(n,k) b_k.$$

By (10), we have

$$\bar{e}_{n,k} = \frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{(n+j)(n-k+j)!}\binom{n-k}{j}\langle x^{n-k+j}\rangle\left(\log\frac{1}{1-x}\right)^j$$

$$= \frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j}{(n+j)(n-k+j)!}\binom{n-k}{j}j!\,s_1(n-k+j,j).$$

By (8), we have

$$b_n = \sum_{k=0}^{n}\frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j j!}{(n+j)(n-k+j)!}\binom{n-k}{j}s_1(n-k+j,j)B_k.$$

Therefore, we obtain the following inverse relation:

$$\begin{cases} B_n = \sum_{k=0}^{\infty} s_1(n,k)b_k, \\ b_n = \sum_{k=0}^{n}\frac{n!}{(k-1)!}\binom{2n-k}{n}\sum_{j=1}^{n-k}\frac{(-1)^j j!}{(n+j)(n-k+j)!}\binom{n-k}{j}s_1(n-k+j,j)B_k. \end{cases} \qquad \square$$

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