ON LUCAS ν-TRIANGLES

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1. INTRODUCTION

Let $\mathbb{N}=\{0,1,2,\dots\}$ and $\mathbb{Z}^+=\mathbb{N}\setminus\{0\}$. Let A and B be fixed nonzero integers with $(A,B)=1$, and write $\Delta=A^2-4B$. We will assume $\Delta\neq 0$, which excludes degenerate cases including $|A|=2$ and $B=1$. Define $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ as follows:

$$u_0=0,\ u_1=1 \text{ and } u_{n+1}=Au_n-Bu_{n-1} \text{ for } n\in\mathbb{Z}^+; \tag{1.1}$$

$$v_0=2,\ v_1=A \text{ and } v_{n+1}=Av_n-Bv_{n-1} \text{ for } n\in\mathbb{Z}^+. \tag{1.2}$$

They are called Lucas sequences. The addition formulas

$$u_{m+n}=\frac{u_mv_n+u_nv_m}{2} \text{ and } v_{m+n}=\frac{v_mv_n+\Delta u_mu_n}{2} \text{ for } m,n\in\mathbb{N} \tag{1.3}$$

are well known. A list of such basic identities can be found in [3].

If $A\neq\pm1$ or $B\neq1$, then $u_1,u_2,\dots$ are nonzero by [1], and so are $v_1=u_2/u_1$, $v_2=u_4/u_2$, ... In the case $A^2=B=1$, we noted in [1] that $u_n=0\Leftrightarrow 3\mid n$. If $v_n=0$, then $u_{2n}=u_nv_n=0$; hence, $3\mid n$ and $u_n=0$, which is impossible since $v_n^2-\Delta u_n^2=4B^n$ (cf. [3]). Thus, $v_0,v_1,v_2,\dots$ are all nonzero.

We set $v_n!=\prod_{0<k\le n}v_k$ for $n\in\mathbb{N}$, and regard an empty product as value 1. For $n,k\in\mathbb{N}$ with $n\ge k$, we define the *Lucas v-triangle* $\left\{{n \atop k}\right\}$ as follows:

$$\left\{{n \atop k}\right\}=\frac{v_n!}{v_k!\,v_{n-k}!}. \tag{1.4}$$

(This definition is not new in the case $A=1$ and $B=-1$; the reader may consult Wells [5].) Similarly, in the case $A\neq\pm1$ or $B\neq1$, Lucas u-triangles can be defined in terms of the sequence $\{u_n\}_{n\in\mathbb{N}}$ (cf. [1]).

Let q be a positive integer. Clearly, $v_q\equiv A^q \pmod B$ and hence $(B,v_q)=1$. Let v_q^* denote the largest divisor of v_q prime to $v_0,\dots,v_{q-1}$. Then v_q^* is odd since $v_0=2$. It is known that $(v_m,v_n)\in\{1,2,|v_{(m,n)}|\}$ for $m,n\in\mathbb{N}$ (cf. [3] or (2.21) of [4]). If $q\nmid n$, then $(v_{(q,n)},v_q^*)=1$ and so $(v_q^*,v_n)=(v_q^*,(v_q,v_n))=1$.

For $m\in\mathbb{Z}$, we let $D(m)$ denote the ring of rationals in the form a/b with $a\in\mathbb{Z}$, $b\in\mathbb{Z}^+$, and $(b,m)=1$. When $r\in D(m)$, by $x\equiv r \pmod m$ we mean that x can be written as $r+my$ with $y\in D(m)$. For a positive integer q, if $0\le k\le n<q$ then $\left\{{n \atop k}\right\}$ lies in $D(v_q^*)$.

Let p be a prime. A famous theorem of Lucas concerning Pascal's triangles (i.e., binomial coefficients) states that

$$\binom{mp+s}{np+t}\equiv\binom{m}{n}\binom{s}{t} \pmod p$$

if m, n, s, t are nonnegative integers with $s,t<p$. An analogy to Lucas u-triangles was obtained by Kimball and Webb [2], by Wilson [6] in some special cases, and by Hu and Sun [1] for the general case. In this paper we aim to establish a similar result for Lucas v-triangles. Recall that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is strong divisible, i.e., $(u_m, u_n)=|u_{(m,n)}|$ for all $m,n\in\mathbb{N}$, while $\{v_n\}_{n\in\mathbb{N}}$ is not in general. This makes our goal more challenging.

Our main result is as follows.

Theorem: Let q be a positive integer. For $m,n\in 2\mathbb{N}=\{0,2,4,\dots\}$ with $m\geq n$, and $s,t\in\mathbb{N}$ with $q>s\geq t$, we have

$$\binom{m/2}{n/2}\begin{Bmatrix} mq+s\\ nq+t\end{Bmatrix}\equiv\binom{m}{n}\begin{Bmatrix} s\\ t\end{Bmatrix}(-B^q)^{\frac{m-n}{2}(nq+t)+\frac{n}{2}(s-t)} \pmod{v_q^*}. \tag{1.5}$$

A proof of the theorem will be presented in Section 3; it depends on several lemmas given in the next section. Our method is different from that of [5] and [6].

2. THREE LEMMAS

As usual, for a real number x, we use $\lfloor x\rfloor$ to denote the greatest integer not exceeding x.

Lemma 2.1: Let $k\in\mathbb{Z}^+$ and $q\in\mathbb{N}$. Then

$$u_{kq}=u_q\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k-i-1}{i}v_q^{k-1-2i}(-B^q)^i \tag{2.1}$$

and

$$v_{kq}=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\frac{k}{k-i}\binom{k-i}{i}v_q^{k-2i}(-B^q)^i, \tag{2.2}$$

where

$$\frac{k}{k-i}\binom{k-i}{i}\in\mathbb{Z}\ \text{ for } i=0,1,\dots,\left\lfloor\frac{k}{2}\right\rfloor.$$

This known result was included in [3].

From Lemma 2.1, we can deduce

Lemma 2.2: Let $k,q,r\in\mathbb{N}$. Then

$$2v_{kq+r}\equiv\begin{cases}2v_r(-B^q)^{k/2}+\frac{k}{2}(-B^q)^{k/2-1}\Delta u_q u_r v_q \pmod{v_q^2} & \text{if } 2\mid k,\\ \Delta u_q u_r(-B^q)^{(k-1)/2}+k(-B^q)^{(k-1)/2}v_r v_q \pmod{v_q^2} & \text{if } 2\nmid k.\end{cases} \tag{2.3}$$

Moreover, providing $2\nmid k$, we have

$$\frac{v_{kq}}{k}\equiv(-B^q)^{(k-1)/2}v_q \pmod{v_q^2}. \tag{2.4}$$

Proof: The case $k=0$ is trivial. Below we let $k\in\mathbb{Z}^+$. Obviously,

$$\binom{k-1-\lfloor\frac{k-1}{2}\rfloor}{\lfloor\frac{k-1}{2}\rfloor}v_q^{k-1-2\lfloor\frac{k-1}{2}\rfloor}(-B^q)^{\lfloor\frac{k-1}{2}\rfloor}=\begin{cases}\frac{k}{2}(-B^q)^{k/2-1}v_q & \text{if } 2\mid k,\\ (-B^q)^{(k-1)/2} & \text{if } 2\nmid k.\end{cases}$$

So, by (2.1), we have

$$u_{kq} \equiv u_q \times \begin{cases} \frac{k}{2}(-B^q)^{k/2-1}v_q \pmod{v_q^2} & \text{if } 2\mid k, \\ (-B^q)^{(k-1)/2} \pmod{v_q^2} & \text{if } 2\nmid k. \end{cases}$$

Similarly, (2.2) implies that

$$\begin{aligned} v_{kq} &\equiv \frac{k}{k-\lfloor\frac{k}{2}\rfloor}\binom{k-\lfloor\frac{k}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor} v_q^{k-2\lfloor\frac{k}{2}\rfloor}(-B^q)^{\lfloor\frac{k}{2}\rfloor} \\ &\equiv \begin{cases} 2(-B^q)^{k/2} \pmod{v_q^2} & \text{if } 2\mid k, \\ k(-B^q)^{(k-1)/2}v_q \pmod{v_q^2} & \text{if } 2\nmid k. \end{cases} \end{aligned}$$

As $2v_{kq+r} = v_{kq}v_r + \Delta u_{kq}u_r$, (2.3) follows from the above.

Now suppose that k is odd. By Lemma 2.1,

$$\begin{aligned} \frac{v_{kq}}{k} &= \sum_{i=0}^{\frac{k-1}{2}} \frac{1}{k-i}\binom{k-i}{k-2i} v_q^{k-2i}(-B^q)^i \\ &= v_q(-B^q)^{(k-1)/2} + v_q^2 \sum_{0\le i\le\frac{k-3}{2}} \frac{v_q^{k-2i-2}}{k-2i}\binom{k-i-1}{k-2i-1}(-B^q)^i. \end{aligned}$$

For any prime p, clearly $p^{3-2}/3 \in D(p)$, and for $n = 4, 5, \ldots$ we also have $p^{n-2}/n \in D(p)$ because

$$(1+p-1)^{n-2} \ge 1+\binom{n-2}{1}(p-1)+(p-1)^{n-2} \ge 2+(n-2)(p-1) \ge n.$$

When $0 \le i \le (k-3)/2$, by the above, $v_q^{k-2i-2}/(k-2i) \in D(p)$ for any prime p dividing v_q, so $v_q^{k-2i-2}/(k-2i) \in D(v_q)$. Thus, we have the desired (2.4).

Lemma 2.3: Let q be any positive integer, and let m, n be even integers with $m \ge n \ge 0$. Then

$$\binom{m/2}{n/2}\begin{Bmatrix} mq \\ nq \end{Bmatrix} \equiv \binom{m}{n}(-B^q)^{\frac{m-n}{2}nq} \pmod{v_q^*}. \tag{2.5}$$

Proof: Recall that $(v_q^*, 2B) = 1$. In view of (2.4), for $i = 1, 3, 5, \ldots$ we have

$$\frac{v_{iq}}{i} \equiv (-B^q)^{\frac{i-1}{2}} v_q \pmod{v_q^2}.$$

Observe that

$$\begin{aligned} \binom{m/2}{n/2}\prod_{\substack{0\le k<n\\ 2\nmid k}} \frac{v_{(m-k)q}}{v_{(n-k)q}} &= \prod_{0\le j<n/2}\frac{m/2-j}{n/2-j}\cdot\prod_{\substack{0\le k<n\\ 2\nmid k}}\frac{m-k}{n-k}\cdot\prod_{\substack{0\le k<n\\ 2\nmid k}}\frac{v_{(m-k)q}/(m-k)}{v_{(n-k)q}/(n-k)} \\ &= \prod_{0\le k<n}\frac{m-k}{n-k}\cdot\prod_{\substack{0\le k<n\\ 2\nmid k}}\frac{v_{(m-k)q}/((m-k)v_q)}{v_{(n-k)q}/((n-k)v_q)} \\ &\equiv \binom{m}{n}\prod_{\substack{0\le k<n\\ 2\nmid k}}\frac{(-B^q)^{(m-k-1)/2}}{(-B^q)^{(n-k-1)/2}} = \binom{m}{n}(-B^q)^{\frac{m-n}{2}\cdot\frac{n}{2}} \pmod{v_q^*}. \end{aligned}$$

By (2.2), for $i = 2, 4, 6, \ldots$ we have $v_{iq} \equiv 2(-B^q)^{i/2} \pmod{v_q^2}$, and hence $(v_{iq}, v_q^*) = 1$.

Whenever $0 \le j < nq$ and $j \not\equiv q \pmod{2q}$, we have $(v_{nq-j}, v_q^*) = 1$. Also,

$$2v_{mq-j} = 2v_{(m-n)q+(nq-j)} \equiv 2v_{nq-j}(-B^q)^{(m-n)/2} \pmod{v_q^*}$$

by (2.3). Thus,

$$\prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} \frac{v_{mq-j}}{v_{nq-j}} \equiv \prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} (-B^q)^{\frac{m-n}{2}} = (-B^q)^{\frac{m-n}{2}(nq-\frac{n}{2})} \pmod{v_q^*}.$$

Combining the above, we obtain that

$$\begin{aligned}
\binom{m/2}{n/2}\left\{\begin{matrix} mq \\ nq \end{matrix}\right\} &= \binom{m/2}{n/2} \prod_{0 \le j < nq} \frac{v_{mq-j}}{v_{nq-j}} \\
&= \binom{m/2}{n/2} \prod_{\substack{0 \le k < n \\ 2 \nmid k}} \frac{v_{(m-k)q}}{v_{(n-k)q}} \cdot \prod_{\substack{0 \le j < nq \\ 2q \nmid j-q}} \frac{v_{mq-j}}{v_{nq-j}} \\
&\equiv \binom{m}{n} (-B^q)^{\frac{m-n}{2}\cdot\frac{n}{2}} (-B^q)^{\frac{m-n}{2}(nq-\frac{n}{2})} \\
&= \binom{m}{n} (-B^q)^{\frac{m-n}{2}nq} \pmod{v_q^*}.
\end{aligned}$$

This completes the proof of Lemma 2.3.

3. PROOF OF THE THEOREM

Recall that

$$\left\{\begin{matrix} s \\ t \end{matrix}\right\} \in D(v_q^*)$$

since $s < q$. Clearly,

$$\left\{\begin{matrix} mq+s \\ nq+t \end{matrix}\right\} = \frac{\prod_{(m-n)q < j \le mq} v_j}{\prod_{0 < j \le nq} v_j} \cdot \frac{\prod_{0 < r \le s}(2v_{mq+r})}{\prod_{0 < r \le t}(2v_{nq+r}) \cdot \prod_{0 < r \le s-t}(2v_{(m-n)q+r})}.$$

Applying Lemmas 2.2 and 2.3, we then get that

$$\begin{aligned}
\binom{m/2}{n/2}\left\{\begin{matrix} mq+s \\ nq+t \end{matrix}\right\} &\equiv \binom{m/2}{n/2}\left\{\begin{matrix} mq \\ nq \end{matrix}\right\} \frac{\prod_{0 < r \le s}(2v_r(-B^q)^{m/2})}{\prod_{0 < r \le t}(2v_r(-B^q)^{n/2}) \cdot \prod_{0 < r \le s-t}(2v_r(-B^q)^{(m-n)/2})} \\
&\equiv \binom{m}{n} (-B^q)^{\frac{m-n}{2}nq} \frac{v_s!}{v_t!\, v_{s-t}!} (-B^q)^{\frac{m}{2}s - \frac{n}{2}t - \frac{m-n}{2}(s-t)} \\
&\equiv \binom{m}{n} \left\{\begin{matrix} s \\ t \end{matrix}\right\} (-B^q)^{\frac{m-n}{2}(nq+t) + \frac{n}{2}(s-t)} \pmod{v_q^*}.
\end{aligned}$$

This completes the proof of the Theorem.

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