HEPTAGONAL NUMBERS IN THE LUCAS SEQUENCE AND DIOPHANTINE EQUATIONS $x^2(5x-3)^2 = 20y^2 \pm 16$

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1. INTRODUCTION

The numbers of the form $\frac{m(5m-3)}{2}$, where m is any positive integer, are called *heptagonal numbers*. That is, 1, 7, 18, 34, 55, 81, ..., listed in [4] as sequence number 1826. In this paper, it is established that 1, 4, 7, and 18 are the only generalized heptagonal numbers (where m is any integer) in the *Lucas sequence* $\{L_n\}$. As a result, the Diophantine equations of the title are solved. Earlier, Cohn [1] identified the squares (listed in [4] as sequence number 1340) and Luo (see [2] and [3]) identified the triangular and pentagonal numbers (listed in [4] as sequence numbers 1002 and 1562, respectively) in $\{L_n\}$.

2. IDENTITIES AND PRELIMINARY LEMMAS

We have the following well-known properties of $\{L_n\}$ and $\{F_n\}$:

$$L_{-n} = (-1)^n L_n \quad \text{and} \quad F_{-n} = (-1)^{n+1} F_n; \tag{1}$$

$$2 \mid L_n \text{ iff } 3 \mid n \quad \text{and} \quad 3 \mid L_n \text{ iff } n \equiv 2 \pmod 4; \tag{2}$$

$$L_n^2 = 5F_n^2 + 4(-1)^n. \tag{3}$$

If $m \equiv \pm 2 \pmod 6$, then the congruence

$$L_{n+2km} \equiv (-1)^k L_n \pmod{L_m} \tag{4}$$

holds, where k is an integer.

Since N is generalized heptagonal if and only if $40N+9$ is the square of an integer congruent to 7 (mod 10), we identify those n for which $40L_n + 9$ is a perfect square. We begin with

Lemma 1: Suppose $n \equiv 1, 3, \pm 4$, or $\pm 6 \pmod{18200}$. Then $40L_n + 9$ is a perfect square if and only if $n \equiv 1, 3, \pm 4$, or ± 6.

Proof: To prove this, we adopt the following procedure: Suppose $n \equiv \varepsilon \pmod N$ and $n \neq \varepsilon$. Then n can be written as $n = 2 \cdot \delta \cdot 2^\theta \cdot g + \varepsilon$, where $\theta \geq \gamma$ and $2 \nmid g$. And since, for $\theta \geq \gamma$, $2^{\theta+s} \equiv 2^\theta \pmod p$, taking

$$m = \begin{cases} \mu \cdot 2^\theta & \text{if } \theta \equiv \zeta \pmod s, \\ 2^\theta & \text{otherwise,} \end{cases}$$

we get that

$$m \equiv c \pmod p \quad \text{and} \quad n = 2km + \varepsilon, \text{ where } k \text{ is odd.} \tag{5}$$

Now, by (4), (5), and the fact that $m \equiv \pm 2 \pmod 6$, we have

$$40L_n + 9 = 40L_{2km+\varepsilon} + 9 \equiv 40(-1)^k L_\varepsilon + 9 \pmod{L_m}.$$

Since either m or n is not congruent to 2 modulo 4 we have, by (3), the Jacobi symbol

$$\left(\frac{40L_n+9}{L_m}\right)=\left(\frac{-40L_\varepsilon+9}{L_m}\right)=\left(\frac{L_m}{M}\right). \tag{6}$$

But, modulo M, $\{L_n\}$ is periodic with period P (i.e., $L_{n+Pt} \equiv L_n \pmod{M}$ for all integers $t \geq 0$). Thus, from (1) and (5), we have $\left(\frac{L_m}{M}\right) = -1$. Therefore, by (6), it follows that $\left(\frac{40L_n+9}{L_m}\right) = -1$ for $n \neq \varepsilon$, showing that $40L_n+9$ is not a perfect square. For each value of $n = \varepsilon$, the corresponding values are tabulated in Table A.

TABLE A

<table>
<tr><th>ε</th><th>N</th><th>δ</th><th>γ</th><th>s</th><th>p</th><th>μ</th><th>$\zeta \pmod{s}$</th><th>$c \pmod{p}$</th><th>M</th><th>P</th></tr>
<tr><td>1</td><td>$2^2 \cdot 5$</td><td>5</td><td>1</td><td>4</td><td>30</td><td>5</td><td>2, 3</td><td>2, ±10, 16</td><td>31</td><td>30</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">$2^2 \cdot 5 \cdot 13$</td><td rowspan="2">$5 \cdot 13$</td><td rowspan="2">1</td><td rowspan="2">20</td><td rowspan="2">50</td><td>$5 \cdot 13$</td><td>3, ±5, 9, 13, 19.</td><td rowspan="2">±2, ±4, ±16, ±20, ±22, ±24.</td><td rowspan="2">151</td><td rowspan="2">50</td></tr>
<tr><td>5</td><td>6, 8, 16, 18.</td></tr>
<tr><td rowspan="2">±4</td><td rowspan="2">$2^2 \cdot 5^2$</td><td rowspan="2">5^2</td><td rowspan="2">1</td><td rowspan="2">36</td><td rowspan="2">270</td><td>5^2</td><td>7, 16, 34, 35.</td><td rowspan="2">2, 8, ±20, ±40, 46, 62, 64, ±80, 94, 98, ±110, 122, 124, 130, 136, 152, 166, 182, 212, 218, 226, 244, 256, 260.</td><td rowspan="2">271</td><td rowspan="2">270</td></tr>
<tr><td>5</td><td>2, ±4, ±5, ±9, 10, 11, ±13, 14, 28, 30.</td></tr>
<tr><td rowspan="2">±6</td><td rowspan="2">$2^3 \cdot 5^2 \cdot 7$</td><td rowspan="2">$5^2 \cdot 7$</td><td rowspan="2">2</td><td rowspan="2">12</td><td rowspan="2">156</td><td>5^2</td><td>0, 10.</td><td rowspan="2">4, 8, 16, 64, 80, 100.</td><td rowspan="2">79</td><td rowspan="2">78</td></tr>
<tr><td>5</td><td>±5, 9, 11.</td></tr>
</table>

Since the L.C.M. of $(2^5 \cdot 5, 2^2 \cdot 5 \cdot 13, 2^2 \cdot 5^2, 2^3 \cdot 5^2 \cdot 7) = 18200$, Lemma 1 follows from Table A. □

Lemma 2: $40L_n+9$ is not a perfect square if $n \not\equiv 1, 3, \pm 4$, or $\pm 6 \pmod{18200}$.

Proof: We prove the lemma in different steps, eliminating at each stage certain integers n congruent modulo 18200 for which $40L_n+9$ is not a square. In each step, we choose an integer M such that the period P (of the sequence $\{L_n\}$ mod M) is a divisor of 18200 and thereby eliminate certain residue classes modulo P. We tabulate these in the following way (Table B).

TABLE B

Period P	Modulus M	Required values of n where $\left(\frac{40L_n+9}{m}\right) = -1$	Left out values of n (mod k) where k is a positive integer
10	11	±2, 9.	0, 1, ±3, 4, 5 or 6 (mod 10)
50	101	0, 11, ±15, ±16, 17, ±20, ±24, 27, 43, 45, 47.	1, 3, ±4, ±6, ±10, 13, 21, 23, 25 or 31 (mod 50)
	151	5, 7, ±14, 33, 37, 41.	
100	3001	±10, 13, 21, 23, ±44, 53, 71, 75.	1, 3, ±4, ±6, 25, 31, ±40, ±46, 51, 63, 73 or 81 (mod 100)
14	29	0, 5, 13.	1, 3, ±4, ±6, ±104, ±246, 281, ±340 (mod 700)
28	13	9, ±10, ±12, 15, 17, 21, 23, 25.	
70	71	11, 15, 31, 53, 63.	
	911	±16, ±20.	
700	701	±60, ±106, ±146, ±204, 231, ±254, ±304, ±306, 563, 651.	
350	54601	323	
26	521	0, ±8, ±9, ±10, ±11, ±12, 19.	1, 3, ±4, ±6, ±2346 or 7281 (mod 9100)
52	233	±5, ±20, ±21, ±24, 29, 39, 49.	
130	131	23, ±30, 33, 51, ±54, ±56, 91, 103, 111.	
	24571	53.	
650	3251	±46, ±106, ±154, ±256, ±306.	
910	50051	±386.	
8	3	0, 5, 7.	1, 3, ±4, ±6 (mod 18200)
40	41	±14.	
728	232961	±202.	
1400	28001	281.	

3. MAIN THEOREM

Theorem:

(a) L_n is a generalized heptagonal number only for $n = 1, 3, \pm4$, or ±6.

(b) L_n is a heptagonal number only for $n = 1, \pm4$, or ±6.

Proof:

(a) The first part of the theorem follows from Lemmas 1 and 2.

(b) Since an integer N is heptagonal if and only if $40N+9 = (10m-3)^2$, where m is a positive integer, we have the following table. □

TABLE C

n	1	3	±4	±6
L_n	1	4	7	18
$40L_n+9$	7^2	13^2	17^2	27^2
m	1	-1	2	3
F_n	1	2	±3	±8

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

It is well known that if $x_1 + y_1\sqrt{D}$ (where D is not a perfect square, x_1, y_1 are least positive integers) is the fundamental solution of Pell's equation $x^2 - Dy^2 = \pm 1$, then the general solution is given by $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$. Therefore, by (3), it follows that

$$L_{2n} + \sqrt{5}F_{2n} \text{ is a solution of } x^2 - 5y^2 = 4, \tag{7}$$

while

$$L_{2n+1} + \sqrt{5}F_{2n+1} \text{ is a solution of } x^2 - 5y^2 = -4. \tag{8}$$

We have the following two corollaries.

Corollary 1: The solution set of the Diophantine equation

$$x^2(5x-3)^2 = 20y^2 - 16 \tag{9}$$

is $\{(1, \pm 1), (-1, \pm 2)\}$.

Proof: Writing $X = x(5x-3)/2$, equation (9) reduces to the form

$$X^2 = 5y^2 - 4 \tag{10}$$

whose solutions are, by (8), $L_{2n+1} + \sqrt{5}F_{2n+1}$ for any integer n.

Now $x = m$, $y = b$ is a solution of (9) $\Leftrightarrow \frac{m(5m-3)}{2} + \sqrt{5}b$ is a solution of (10) and the corollary follows from Theorem 1(a) and Table C. □

Similarly, we can prove the following.

Corollary 2: The solution set of the Diophantine equation

$$x^2(5x-3)^2 = 20y^2 + 16$$

is $\{(2, \pm 3), (3, \pm 8)\}$.

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