FIBONACCI TREE IS CRITICALLY BALANCED–A NOTE*

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(*Submitted September 2000-Final Revision February 2001*)

1. INTRODUCTION

To continue a previous note [2] (also [3]) on the morphology of self-similar trees, we reconsider, as simple model trees (see [2] for motivations), the sequence of binary trees $S_k = S_k(a,b)$, $k = 1, 2, \ldots,$ defined recursively for relatively prime integers a, b such that $1 \le a \le b : S_1, \ldots, S_b$ are just one-leaf trees, and, for $k \ge b+1$, the left subtree of S_k is given by S_{k-a} and the right by S_{k-b}. Put $c = \frac{b}{a}$. When $c = 2$, we have $S_k(1,2)$, the Fibonacci tree (of order k).

Denote the number of leaves in S_k by $n_k = n_k(c)$ and write

$$\begin{cases} \lambda_k = \lambda_k(c) = \frac{n_{k-a}}{n_k} \ (k \ge b+1), \\ \lambda = \lambda(c) = \lim\limits_{k\to\infty} \lambda_k, \end{cases}$$

then $\lambda_k : (1-\lambda_k)$ may be considered as a left-to-right weight-proportion in S_k.

The average path length $L_k = L_k(c)$ (i.e., the average number of branchings along the path from the root to a leaf) of S_k is the sum of the lengths of all the paths from the root to leaves divided by n_k.

In Section 2 we show the following relation:

$$G(c)H(c) = 1,$$

where

$$\begin{cases} G(c) = \lim\limits_{k\to\infty} \frac{L_k}{\log n_k}, \\ H(c) = -\lambda \log \lambda - (1-\lambda)\log(1-\lambda). \end{cases}$$

("log" is to the base 2, while "ln" is to the base e.)

That is, we show that the normalized L_k, $L_k / \log n_k$, converges and the limit equals $(H(c))^{-1}$, the inverse of the entropy of the distribution $\lambda, 1-\lambda$. Roughly, $G(c)$ and $H(c)$ express the asymptotic growth and breadth indices, respectively, of the tree.

We will then observe in Section 3 some simple balance properties of S_k and show that the c maximizing $G(c)$ but maintaining S_k balanced for every k is equal to 2.

2. A LIMITING RELATION

The following lemma was implicitly shown in [2] and will be used in the sequel.

Lemma 1:

(a) $\lambda^b = (1-\lambda)^a$;

(b) $\lambda = \lambda(c)$ $(1 \le c)$ is less than 1 and strictly monotone increasing, and $\lambda(1) = \frac{1}{2}$, $\lambda(2) = \frac{\sqrt{5}-1}{2}$;

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* This paper was presented at the Ninth International Conference on Fibonacci Numbers and Their Applications, July 17-22, 2000, Institut Supérieur de Technologie, Luxembourg.

(c) $\frac{1}{k}\log n_k \to \frac{1}{a}(-\log\lambda)$ as $k\to\infty$;

(d) $|\lambda_k - \lambda| \to 0$ exponentially fast as $k\to\infty$.

Theorem 1: $G(c)H(c)=1$.

Proof: It is easy to see that the recursive structure of S_k implies

$$L_k = \lambda_k L_{k-a} + (1-\lambda_k)L_{k-b} + 1 \quad (k \ge b+1) \tag{1}$$

$(L_1 = \cdots = L_b = 0)$, which we are going to compare with the following equation with constant coefficients:

$$x_k = \lambda x_{k-a} + (1-\lambda)x_{k-b} + 1 \quad (k \ge b+1) \tag{2}$$

$(x_1 = \cdots = x_b = 0)$.

Remark: Kapoor and Reingold [4] treated, in a different way, a general recurrence, including (1), derived from the binary trees with costs a and b on the left and right branches.

The characteristic equation $\lambda t^{-a} + (1-\lambda)t^{-b} = 1$ of the homogeneous

$$y_k = \lambda y_{k-a} + (1-\lambda)y_{k-b} \tag{3}$$

clearly has root 1, and it can be shown that $|\alpha| < 1$ for every other root α. Therefore, the general solution of (3) is given by $y_k = C_1 + \varepsilon_k$, where C_1 is a constant and $\varepsilon_k \to 0$ $(k\to\infty)$.

As a particular solution of (2), we have

$$x_k = \frac{(-\log\lambda)}{aH(c)}k \quad (k \ge 1).$$

In fact, the right-hand side of (2) then becomes

$$\begin{aligned}
&\lambda\frac{(-\log\lambda)}{aH(c)}(k-a) + (1-\lambda)\frac{(-\log\lambda)}{aH(c)}(k-b)+1 \\
&= \frac{(-\log\lambda)}{aH(c)}k + \frac{1}{aH(c)}(a\lambda\log\lambda + b(1-\lambda)\log\lambda - a\lambda\log\lambda - a(1-\lambda)\log(1-\lambda)) \\
&= \frac{(-\log\lambda)}{aH(c)}k + \frac{1-\lambda}{aH(c)}\log\left\{\frac{\lambda^b}{(1-\lambda)^a}\right\} = \frac{(-\log\lambda)}{aH(c)}k \quad \text{[by Lemma 1(a)]} \\
&= x_k.
\end{aligned}$$

The solution of (2) is therefore given by

$$x_k = \frac{(-\log\lambda)}{aH(c)}k + C_1 + \varepsilon_k, \tag{4}$$

which we regard as the solution satisfying the initial condition $x_1 = \cdots = x_k = 0$.

Subtract (2) from (1) to get

$$L_k - x_k = \lambda_k(L_{k-a} - x_{k-a}) + (1-\lambda_k)(L_{k-b} - x_{k-b}) + (\lambda_k - \lambda)(x_{k-a} - x_{k-b}),$$

then

$$|L_k - x_k| \le \lambda_k|L_{k-a} - x_{k-a}| + (1-\lambda_k)|L_{k-b} - x_{k-b}| + C_2|\lambda_k - \lambda|, \tag{5}$$

since we can write $|x_{k-a} - x_{k-b}| \le C_2$ from (4).

Now we prove by induction on k that

$$|L_k - x_k| \le C_3 \ln k \quad (k \ge 1) \tag{6}$$

for some constant C_3. Trivially true for $k = 1, \ldots, b$, since $L_k = x_k = 0$ for those k. Suppose $k \ge b+1$, then $\frac{a}{k} \le \frac{b}{k} < 1$. By the induction hypothesis, (5), and the inequality $\ln(1-x) \le x$, we have

$$\begin{aligned}
|L_k - x_k| &\le C_3 \lambda_k \ln(k-a) + C_3(1-\lambda_k)\ln(k-b) + C_2 |\lambda_k - \lambda| \\
&= C_3\left\{\lambda_k\left(\ln k + \ln\left(1-\frac{a}{k}\right)\right) + (1-\lambda_k)\left(\ln k + \ln\left(\cdot 1 - \frac{b}{k}\right)\right)\right\} + C_2 |\lambda_k - \lambda| \\
&\le C_3\left\{\ln k - \frac{1}{k}(a\lambda_k + b(1-\lambda_k))\right\} + C_2 |\lambda_k - \lambda| \\
&\le C_3 \ln k - \frac{aC_3}{k} + C_2 |\lambda_k - \lambda| \le C_3 \ln k,
\end{aligned}$$

where the last inequality holds because, by Lemma 1(d), we could have chosen C_3 large enough so that $-\frac{aC_3}{k} + C_2 |\lambda_k - \lambda| \le 0$ for $k \ge b+1$.

From (4) and (6), we obtain

$$\left| L_k - \frac{(-\log \lambda)}{aH(c)} k - C_1 - \varepsilon_k \right| \le C_3 \ln k;$$

hence,

$$\left| \frac{L_k}{\log n_k} - \frac{1}{H(c)} \frac{(-\log \lambda)}{a} \frac{k}{\log n_k} - \frac{C_1 + \varepsilon_k}{\log n_k} \right| \le C_3 \left(\frac{k}{\log n_k}\right)\left(\frac{\ln k}{k}\right).$$

Therefore, $\frac{L_k}{\log n_k} \to \frac{1}{H(c)}$ $(k \to \infty)$ by Lemma 1(c). □

3. CRITICAL BALANCE

A most pleasing, though rather vague, concept concerning the form of a tree might be the concept of being "balanced as a whole."

One natural definition of "balancedness" (let us call it "w-balanced") of the trees S_k is:

$\{S_k\}$ is said to be *w-balanced* if $n_k \ge n_{k-a} + n_{k-2a}$ for every $k \ge b + a + 1$ (see [2]).

(***Remark:*** $b + a + 1$ is the minimum k such that $n_k \ge 3$.)

Note that the definition takes this form to refer to the *sequence* $\{S_k\}$ not to *individual* S_k for reason of compactness. Also note that the definition may be viewed as stemming from the fact that the condition $n_k \ge n_{k-a} + n_{k-2a}$ can be written as

$$n_{k-a} - (n_k - n_{k-a}) \le (n_k - n_{k-2a}) - n_{k-2a},$$

meaning that the division $n_{k-a} : (n_k - n_{k-a})$ of n_k is balanced better than or equally to the division $n_{k-2a} : (n_k - n_{k-2a})$.

Another pretty concept of balancedness of a binary tree is due to Adelson-Velskii and Landis [1]. Denote the height of S_k by $h_k = h_k(c)$, then their definition adapted to S_k is:

$\{S_k\}$ is said to be *h-balanced* if $h_{k-a} - h_{k-b} \le 1$ for every $k \ge b + a + 1$.

We know from [2] that $h_k = \lceil \frac{k-b}{a} \rceil$ $(k \ge b)$.

It should be mentioned here that, according to Nievergelt and Wong [5], $\{S_k\}$ may be called "α-balanced" $(0 < \alpha \le \frac{1}{2})$ if $\frac{n_{k-b}}{n_k} \ge \alpha$ holds for every $k \ge b+a+1$ and they showed that

$$\left(\frac{L_k}{\log n_k}\right)(-\alpha \log \alpha - (1-\alpha)\log(1-\alpha)) \le 1$$

for α-balanced $\{S_k\}$ [in place of $G(c)H(c) = 1$].

Lemma 2:

(a) $\{S_k\}$ is w-balanced if and only if $c \le 2$.

(b) $\{S_k\}$ is h-balanced if and only if $c \le 2$.

(c) $n_k = n_{k-b} + n_{k-2a}$ for every $k \ge b+a+1$ if and only if $c = 2$.

(d) $h_{k-a} - h_{k-b} = 1$ for every $k \ge b+a+1$ if and only if $c = 2$.

Proof: The proof is simple, comprising the following pieces 1~5.

1. We first note that $n_k = n_{k-a} + n_{k-b}$, and hence the "if" part of (c) is obvious.

2. There are (infinitely) many i such that $n_i < n_{i+1}$. So, if $c < 2$ (i.e., $b < 2a$), we have $n_{k-2a} < n_{k-b}$ for (infinitely) many k, and if $c > 2$ (i.e., $b > 2a$), we have $n_{k-2a} > n_{k-b}$ for (infinitely) many k. This proves the "only if" parts of (a) and (c). An alternative proof is: Divide both sides of $n_k \ge n_{k-a} + n_{k-2a}$ by n_k to obtain

$$1 \ge \left(\frac{n_{k-a}}{n_k}\right) + \left(\frac{n_{k-a}}{n_k}\right)\left(\frac{n_{k-2a}}{n_{k-a}}\right).$$

Let $k \to \infty$, then $1 \ge \lambda(c) + (\lambda(c))^2$. Therefore, we deduce $\lambda(c) \le \frac{\sqrt{5}-1}{2}$, and using Lemma 1(b) finishes the proof of those parts.

3. Proof of the "if" part of (a). Suppose $k \ge b+a+1$. Since $b \le 2a$ by $c \le 2$, we have $n_{k-b} \ge n_{k-2a}$. Hence, $\{S_k\}$ is w-balanced.

4. Suppose $c < 2$. Then $b \le 2a-1$. Take $k = b+ia$ $(i \ge 2)$ to see that

$$\begin{aligned} 0 \le h_{k-a} - h_{k-b} &= \left\lceil \frac{(k-a)-b}{a} \right\rceil - \left\lceil \frac{(k-b)-b}{a} \right\rceil = (i-1) - \left\lceil \frac{ia-b}{a} \right\rceil \\ &\le (i-1) - \left\lceil \frac{ia-(2a-1)}{a} \right\rceil = (i-1)-(i-2) - \left\lceil \frac{1}{a} \right\rceil = 0. \end{aligned}$$

That is, $h_{k-a} - h_{k-b} = 0$ holds for (infinitely) many k.

Suppose $c > 2$. Then $b \ge 2a+1$. In this case, taking $k = b+ia+1$ $(i \ge 2)$ leads us to $h_{k-a} - h_{k-b} = i-(i-2) = 2$. That is, $h_{k-a} - h_{k-b} = 2$ holds for (infinitely) many k.

The two remarks above prove the "only if" parts of (b) and (d).

5. Proof of the "if" parts of (b) and (d). Suppose $b+a+1 \le k \le b+2a$. Then, since $b+1 \le k-a \le b+a$, we have $h_{k-a} - h_{k-b}$ $(= 1-0$ or $1-1) \le 1$. (Furthermore, if $c = 2$, then $k-b \le b$ and $h_{k-a} - h_{k-b} = 1-0 = 1$.)

Suppose next that $k \ge b+2a+1$. From $b \le 2a$, we have

$$\frac{(k-a)-b}{a} \leq \frac{(k-b)-b}{a}+1,$$

and hence, by noting that $k-b \geq 2a+1 \geq b+1$, we have $h_{k-a} \leq h_{k-b}+1$. Therefore, $\{S_k\}$ is h-balanced. (Furthermore, if $c=2$, then $h_{k-a}=h_{k-b}+1$.) $\square$

The (asymptotic) average growth function $G(c)$ is strictly monotone increasing because the entropy $H(c)$ is strictly monotone decreasing. Therefore, the c maximizing $G(c)$ while keeping the S_k balanced for every k equals 2.

SUMMARY

Summarizing, we may say that the Fibonacci tree is critically balanced, and in this sense the Golden-cut point $\lambda(2)$ might be interpreted as the critical balancing point.

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AMS Classification Numbers: 05C05, 11B39, 92C15

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