REFLECTIONS ON THE LAMBDA TRIANGLE

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1. INTRODUCTION

Kappraff [2] described the panels in the pavement of the Reading Room of the Library on the second floor of the San Lorenzo church complex in Florence. Work on the library was begun in 1523 by Pope Clement VII, Giulio di Medici, as a monument to his uncle, Lorenzo di Medici. The library was one of the few successes of Clement's disastrous reign, characterized as it was by bad political decisions (see [1], [11]). In the Timaeus panel of the library, Michelangelo, the designer of the library, used the number relations (the scale) of the lambda figure which had previously been used as the musical system studied by Pythagoras [4].

Kappraff used the *lambda triangle* in Table 1 "found in Plato's *Timaeus* and referred to there as the World Soul." Strictly speaking, the lambda diagram displayed in Table 1 is that given in Taylor [10] but with the empty space between the two slanting lines $\wedge$ (hence the designation *lambda*) filled in a methodical and obvious way. Plato himself does not appear to have used the lambda figure as such though he used the two generating scales 1, 2, 4, 8 and 1, 3, 9, 27 shown by the slanting lines to describe the creation by the Demiurge of the World Soul. These scales are represented linearly (essentially in one line) in the commentary on the *Timaeus* [5].

TABLE 1. The Lambda Triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & \wedge & 3 & & \\ & 4 & / & 6 & \backslash & 9 & \\ 8 & / & 12 & & 18 & \backslash & 27 \end{array}$$

The formation is obvious and one cannot resist the temptation to portray the associated left- and right-triangular arrays (Tables 2 and 3). Clearly, these arrays may be extended infinitely.

TABLE 2. Left-Triangular Lambda Array

1	0	0	0
2	3	0	0
4	6	9	0
8	12	18	27

TABLE 3. Right-Triangular Lambda Array

0	0	0	1
0	0	2	3
0	4	6	9
8	12	18	27

It is the purpose of this paper to describe some of the properties of these arrays and triangles.

2. LAMBDA TRIANGLES

The elements, $u_{n,m}$, of the left-triangular array satisfy the partial difference equation

$$u_{n,m} = u_{n,m-1} + u_{n-1,m-1}, \quad n > 0,\ 0 < m \le n, \tag{2.1}$$

with boundary conditions $u_{n,0} = 2^{n-1}$, $u_{n,m} = 0$ when $n < 0$ and $m > n$, and general term

$$u_{n,m} = 2^{n-m}3^{m-1}, \tag{2.2}$$

where n, m represent the rows and columns, respectively. We can see that the row sums, 1, 5, 19, 65, 211, ... (Sequence M3887 of [8]), are given by the second-order homogeneous linear recurrence relation

$$\begin{aligned} v_n &= 5v_{n-1} - 6v_{n-2}, \quad n \geq 3, \; v_1 = 1, v_2 = 5, \\ &= 3^n - 2^n, \quad n \geq 1. \end{aligned} \tag{2.3}$$

The partial column sums are displayed in Table 4.

TABLE 4. Partial Column Sums of Left-Triangular Lambda Array

1					
3	3				
7	9	9			
15	21	27	27		
31	45	63	81	81	
63	93	135	189	243	243

The elements in the cells of Table 4 satisfy the partial recurrence relation

$$w_{n,m} = w_{n,m-1} + w_{n-1,m} - w_{n-2,m-1}, \quad n \geq m > 1, \tag{2.4}$$

with general term

$$w_{n,m} = 3^{m-1}(2^{n-m+1} - 1). \tag{2.5}$$

We now develop more general properties by means of the polynomials associated with the numbers in lambda triangles.

3. ABSTRACT LAMBDA TRIANGLES

Kappraff's array (Table 1) may be readily abstracted and extended as in Table 5 (a, b integers > 0):

TABLE 5. Abstract Lambda Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & a & & b & & & \\ & & a^2 & & ab & & b^2 & & \\ & a^3 & & a^2b & & ab^2 & & b^3 & \\ a^4 & & a^3b & & a^2b^2 & & ab^3 & & b^4 \\ & & & & \dots & & & & \end{array}$$

The *abstract lambda polynomials* $\mathscr{L}_m(x)$ (where $\mathscr{L}_1(x) = 1$) may be easily read off from the rows of Table 5. To illustrate the situation we have

$$\mathscr{L}_5(x) = a^4 + a^3bx + a^2b^2x^2 + ab^3x^3 + b^4x^4 = \frac{a^5 - b^5x^5}{a - bx}.$$

Interchanging a and b, we get the *abstract reciprocal lambda polynomials* $l_m(x)$ (with $l_1(x) = 1$).

Recurrence relations are, respectively,

$$\mathscr{L}_{m+2}(x) = (a + bx)\,\mathscr{L}_{m+1}(x) - abx\,\mathscr{L}_m(x), \tag{3.1}$$

$$l_{m+2}(x) = (b+ax)l_{m+1}(x) - abxl_m(x). \tag{3.2}$$

Generating functions are, respectively,

$$\sum_{m=1}^{\infty} \mathscr{L}_m(x)y^{m-1} = \{1-[(a+bx)y - abxy^2]\}^{-1}, \tag{3.3}$$

$$\sum_{m=1}^{\infty} l_m(x)y^{m-1} = \{1-[b+ax)y - abxy^2]\}^{-1}. \tag{3.4}$$

Properties of these polynomials may be developed to include, for example:

(i) Other fundamental features such as Binet forms, Simson's formulas, closed forms;

(ii) Convolutions $\mathscr{L}_m^{(k)}(x)$, $l_m^{(k)}(x)$;

(iii) Rising and descending polynomials.

We do this in Section 4 by considering a case closer to the original lambda triangle, namely, when $a = n$, $b = n+1$.

4. GENERALIZED LAMBDA POLYNOMIALS

We consider *generalized lambda polynomials,* $\Lambda_m(x)$, and *reciprocal lambda polynomials,* $\lambda_m(x)$, associated with the *generalized lambda triangle* of Table 6, which should be compared with Table 1.

TABLE 6. Generalized Lambda Triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & n & & n+1 & & & \\
 & & n^2 & & n(n+1) & & (n+1)^2 & & \\
 & n^3 & & n^2(n+1) & & n(n+1)^2 & & (n+1)^3 & \\
n^4 & & n^3(n+1) & & n^2(n+1)^2 & & n(n+1)^3 & & (n+1)^4
\end{array}$$

The two classes of polynomials are related by

$$\begin{aligned} \lambda_m(x) &= x^{m-1}\Lambda_m(\tfrac{1}{x}), \\ \Lambda_m(x) &= x^{m-1}\lambda_m(\tfrac{1}{x}). \end{aligned} \tag{4.1}$$

4.1 $\Lambda_m(x)$ Polynomials

Basic properties of $\Lambda_m(x)$ are listed succinctly hereunder:

$$\begin{aligned}
&\Lambda_0(x) = 0 \\
&\Lambda_1(x) = 1 \\
&\Lambda_2(x) = n+(n+1)x \\
&\Lambda_3(x) = n^2 + n(n+1)x + (n+1)^2x^2 \\
&\Lambda_4(x) = n^3 + n^2(n+1)x + n(n+1)^2x^2 + (n+1)^3x^3 \\
&\Lambda_5(x) = n^4 + n^3(n+1)x + n^2(n+1)^2x^2 + n(n+1)^3x^3 + (n+1)^4x^4 \\
&\cdots
\end{aligned} \tag{4.2}$$

Setting $x = 1$, $m > 0$, we obtain the sequence of coefficient sums, thus (observe the binomial coefficients):

$$\{\Lambda_m(1)\} = \{1, 2n+1, 3n^2+3n+1, 4n^3+6n^2+4n+1, \ldots\}. \tag{4.3}$$

Recurrence relations:

homogeneous:

$$\Lambda_{m+2}(x) = [n+(n+1)x]\Lambda_{m+1}(x) - n(n+1)x\Lambda_m(x). \tag{4.4}$$

inhomoegneous:

$$\Lambda_{m+1}(x) = n\Lambda_m(x) + [(n+1)x]^m \quad (m \geq 0). \tag{4.5}$$

Roots of characteristic equation:

$$n, (n+1)x. \tag{4.6}$$

Closed form:

$$\Lambda_m(x) = \sum_{j=0}^{m-1} n^j[(n+1)x]^{m-1-j}. \tag{4.7}$$

Binet form:

$$\Lambda_m(x) = \frac{[(n+1)x]^m - n^m}{(n+1)x - n}. \tag{4.8}$$

Simson's formula:

$$\Lambda_{m+1}(x)\Lambda_{m-1}(x) - \Lambda_m^2(x) = -[n(n+1)x]^{m-1} \quad (m \geq 1). \tag{4.9}$$

Generating function:

$$\sum_{m=1}^{\infty} \Lambda_m(x)y^{m-1} = \{1-[(n+(n+1)x)y - n(n+1)xy^2]\}^{-1}. \tag{4.10}$$

4.2 Reciprocal $\lambda_m(x)$ Polynomials

$$\begin{aligned}
&\lambda_0(x) = 0\\
&\lambda_1(x) = 1\\
&\lambda_2(x) = (n+1) + nx\\
&\lambda_3(x) = (n+1)^2 + n(n+1)x + n^2x^2\\
&\lambda_4(x) = (n+1)^3 + n(n+1)^2x + n^2(n+1)x^2 + n^3x^3\\
&\lambda_5(x) = (n+1)^4 + n(n+1)^3x + n^2(n+1)^2x^2 + n^3(n+1)x + n^4x^4\\
&\ldots
\end{aligned} \tag{4.11}$$

Setting $x = 1$, $m > 0$, we obtain the sequence of coefficient sums, thus (observe the binomial coefficients):

$$\{\lambda_m(1)\} = \{1, 2n+1, 3n^2+3n+1, 4n^3+6n^2+4n+1, \ldots\} = \{\Lambda_m(1)\}. \tag{4.12}$$

Recurrence relations:

homogeneous:

$$\lambda_{m+2}(x) = [(n+1)+nx]\lambda_{m+1}(x) - n(n+1)x\lambda_m(x). \tag{4.13}$$

inhomogeneous:

$$\lambda_{m+1}(x) = (n+1)\lambda_m(x) + [(n+1)x]^m \quad (m \geq 0). \tag{4.14}$$

Roots of characteristic equation:

$$n+1, nx. \tag{4.15}$$

Closed form:

$$\lambda_m(x) = \sum_{j=0}^{m-1} (n+1)^j [nx]^{m-1-j}. \tag{4.16}$$

Binet form:

$$\lambda_m(x) = \frac{(nx)^m - (n+1)^m}{nx - (n+1)}. \tag{4.17}$$

Simson's formula:

$$\lambda_{m+1}(x)\lambda_{m-1}(x) - \lambda_m^2(x) = -[n(n+1)x]^{m-1} \quad (m \geq 1). \tag{4.18}$$

Generating function:

$$\sum_{m=1}^{\infty} \lambda_m(x) y^{m-1} = \{1 - [(n+1+nx)y - n(n+1)xy^2]\}^{-1}. \tag{4.19}$$

5. RELATED POLYNOMIALS

In this section, polynomial properties of related convolutions and of rising and falling diagonals are sketched.

5.1 Convolutions

There are two types of lambda convolution polynomials which are related by

$$\lambda_m^{(k)}(x) = x^{m-1} \Lambda_m^{(k)}(\tfrac{1}{x}), \tag{5.1}$$

$$\Lambda_m^{(k)}(x) = x^{m-1} \lambda_m^{(k)}(\tfrac{1}{x}), \tag{5.2}$$

in which $\Lambda_m^{(k)}(x)$ and $\lambda_m^{(k)}(x)$ are the k^{th} *convolutions* of $\Lambda_m(x)$ and $\lambda_m(x)$, respectively, and $\Lambda_m^{(k)}(x)$ is defined in terms of a generating function

$$\sum_{m=1}^{\infty} \Lambda_m^{(k)}(x) y^{m-1} = \{1 - [(n+(n+1)x)y - n(n+1)xy^2]\}^{-(k+1)}, \tag{5.3}$$

whence we get the recurrence relation

$$\Lambda_m^{(k)}(x) = \Lambda_m^{(k+1)}(x) - (n+(n+1)x)\Lambda_{m-1}^{(k+1)}(x) + n(n+1)x\Lambda_{m-2}^{(k+1)}(x). \tag{5.4}$$

For instance, when $k = 1$:

$$\begin{aligned}
&\Lambda_0^{(1)}(x) = 0 \ \ \text{(definition)} \\
&\Lambda_1^{(1)}(x) = 1 \\
&\Lambda_2^{(1)}(x) = 2n + 2(n+1)x \\
&\Lambda_3^{(1)}(x) = 3n^2 + 4n(n+1)x + 3(n+1)^2 x^2 \\
&\Lambda_4^{(1)}(x) = 4n^3 + 6n^2(n+1)x + 6n(n+1)^2 x^2 + 4(n+1)^3 x^3
\end{aligned} \tag{5.5}$$

Analogously to (5.3) there is a generating function for $\lambda_m^{(k)}(x)$ with $n \leftrightarrow n+1$.

If we consider $\partial\left(\sum_{m=1}^{\infty} \Lambda_m^{(k)}(x) y^{m-1}\right) / \partial y$, then we get

$$(m-1)\Lambda_m^{(k-1)}(x) = k\{(n+(n+1)x) - 2n(n+1)xy\} \tag{5.6}$$

$$= k\{(n+n(n+1)x)\Lambda_{m-1}^{(k)}(x) - 2n(n+1)x\Lambda_{m-2}^{(k)}(x)\}. \tag{5.7}$$

Replace k by $k-1$ in Equation (5.4):

$$\Lambda_m^{(k-1)}(x) = \Lambda_m^{(k)}(x) - (n+(n+1)x)\Lambda_{m-1}^{(k)}(x) + n(n+1)\Lambda_{m-2}^{(k)}(x). \tag{5.8}$$

Now eliminate $\Lambda_m^{(k)}(x)$ from (5.7) and (5.8) to get the recurrence

$$(m-1)\Lambda_m^{(k)}(x) = [k+m-1](n+(n+1)x)\Lambda_{m-1}^{(k)}(x) - n(n+1)[2k+m-1]\Lambda_{m-2}^{(k)}(x).$$

From this, with $k=1$, $m \to m+1$, we can get

$$m\Lambda_{m+1}^{(1)}(1) = (m+1)(2n+1)\Lambda_m^{(1)}(1) - (m+2)n(n+1)\Lambda_{m-1}^{(1)}(1). \tag{5.9}$$

Let $n=2$ in Equation (5.9). Then

$$m\Lambda_{m+1}^{(1)}(1) = 5(m+1)\Lambda_m^{(1)}(1) - 6(m+2)\Lambda_{m-1}^{(1)}(1). \tag{5.10}$$

Notice that in $\{\Lambda_m^{(1)}(x)\}$ (reference (5.5) above) the numerical coefficients form a neat triangle as displayed in Table 7, in which the row sums are the tetrahedral numbers $\binom{n+3}{3}$ (that is, 1, 4, 10, 20, 35, ...) and the rising diagonal sums belong to Sequence 1349 of [8] with general terms $\frac{1}{4}\binom{n+3}{3}$, n odd, and $n(n+2)(n+4)/24$, n even.

TABLE 7. Lambda Convolution Coefficients

				1				
			2		2			
		3		4		3		
	4		6		6		4	
5		8		9		8		5

5.2 Rising and Descending Polynomials

Denote the *rising and descending polynomials* of $\Lambda_m(x)$ and $\lambda_m(x)$ by $R_m(x)$ and $r_m(x)$ and $D_m(x)$ and $d_m(x)$, respectively. They are related, in each case, by the interchange of n and $n+1$.

$\Lambda_m(x)$ Rising

$$\begin{aligned} R_1(x) &= 1 \\ R_2(x) &= n \\ R_3(x) &= n^2+(n+1)x \\ R_4(x) &= n^3+n(n+1)x \\ R_5(x) &= n^4+n^2(n+1)x+(n+1)^2x^2 \\ R_6(x) &= n^5+n^3(n+1)x+n(n+1)^2x^2 \\ &\dots \end{aligned} \tag{5.11}$$

Setting $n=2$ and $x=1$, we obtain the sequence

$$\{R_m(1)\} = \{1, 2, 7, 14, 37, 74, 175, 350, \dots\}. \tag{5.12}$$

Recurrence relations:

homogeneous:

$$R_{2m+1}(x) = [n^2+(n+1)x]R_{2m-1}(x) - n^2(n+1)xR_{2n-3}(x) \quad (m \geq 2), \tag{5.13}$$

$$R_{2m}(x) = nR_{2n-1}(x) \quad (m \geq 1). \tag{5.14}$$

inhomogeneous:

$$R_{2m+1}(x) = nR_{2m}(x) + ((n+1)x)^m \quad (m \geq 0). \tag{5.15}$$

$\lambda_m(x)$ Rising

$$\begin{aligned} &r_1(x) = 1 \\ &r_2(x) = n+1 \\ &r_3(x) = (n+1)^2 + nx \\ &r_4(x) = (n+1)^3 + n(n+1)x \\ &r_5(x) = (n+1)^4 + n(n+1)^2 x + n^2 x^2 \\ &r_6(x) = (n+1)^5 + n(n+1)^3 x + n^2(n+1)x^2 \\ &\ldots \end{aligned} \tag{5.16}$$

Setting $n = 2$ and $x = 1$, we obtain the sequence

$$\{r_m(1)\} = \{1, 3, 11, 33, 103, 309, 935, \ldots\}. \tag{5.17}$$

Recurrence relations:

homogeneous:

$$r_{2m+1}(x) = [(n+1)^2 + nx]r_{2m-1}(x) - n(n+1)^2 x r_{2m-3}(x) \quad (m \geq 2), \tag{5.18}$$

$$r_{2m}(x) = (n+1)r_{2m-1}(x) \quad (m \geq 1). \tag{5.19}$$

inhomogeneous:

$$r_{2m+1}(x) = (n+1)r_{2m}(x) + (nx)^m \quad (m \geq 0). \tag{5.20}$$

Observe from (5.14) and (5.19) the link

$$(n+1)R_{2m}(x)r_{2m-1}(x) = nR_{2m-1}(x)r_{2m}(x) \quad (m \geq 1). \tag{5.21}$$

A *quasi-reciprocal relationship* between $R_m(x)$ and $r_m(x)$ can be evolved subject to certain provisos regarding n and $n+1$. For example,

$$R_5(x) = x^2 r_5(\tfrac{1}{x}) \text{ if } n^2 \to n,\ n+1 \to (n+1)^2.$$

Check for $r_5(x)$ and $R_5(\frac{1}{x})$. Likewise, look at $R_6(x)$ and $r_6(\frac{1}{x})$, and $r_6(x)$ and $R_6(\frac{1}{x})$.

Patterns for m odd and m even emerge.

$\Lambda_m(x)$ Descending

Clearly, $D_m(x) = n^{m-1}(1-(n+1)x)^{-1}$, so

$$D_m(x) = nD_{m-1}(x) \tag{5.22}$$

and

$$\frac{dD_m(x)}{dx} = (n+1)n^{m-1}(1-(n+1)x)^{-2}. \tag{5.23}$$

If

$$D \equiv D(x, y) \equiv \sum_{m=1}^{\infty} D_m(x)y^{m-1} = (1-(n+1)xy)^{-1},$$

then

$$\frac{\partial D / \partial y}{\partial D / \partial x} = \frac{x}{y}. \tag{5.24}$$

$\lambda_m(x)$ Descending

Obviously, $d_m(x) = (n+1)^{m-1}(1-nx)^{-1}$, so

$$d_m(x) = (n+1)d_{m-1}(x)$$

and

$$\frac{dd_m(x)}{dx} = n(n+1)^{m-1}(1-nx)^{-2}.$$

If

$$d \equiv d(x, y) \equiv \sum_{n=1}^{\infty} d_m(x)y^{m-1} = (1-nxy)^{-1},$$

then

$$\frac{\partial d / \partial y}{\partial d / \partial x} = \frac{x}{y}.$$

Hence,

$$\frac{\partial D}{\partial x}\frac{\partial d}{\partial y} = \frac{\partial D}{\partial y}\frac{\partial d}{\partial x}$$

and

$$\frac{\frac{\partial D_m(x)}{\partial x}}{\frac{\partial d_m(x)}{\partial x}} = \left(\frac{n}{n+1}\right)^{m-1}\left[\frac{1-nx}{1-(n+1)x}\right]^2. \tag{5.25}$$

Special Case

Putting $n = 2$ in the results of Sections 4 and 5, we obtain the particular cases for the original configuration in Table 1.

Further investigation of rising and descending polynomials could be undertaken; for example, the establishment of closed summation forms for $R_m(x)$ and $r_m(x)$.

6. FIBONACCI-LAMBDA TRIANGLES

6.1 Fibonacci-Lambda Polynomials

Suppose now that we replace a and b in Section 3 by α and β, respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We then have a triangle whose row sums are, successively,

$$\begin{aligned} 1 &= \frac{\alpha^1-\beta^1}{\alpha-\beta} = F_1 \\ \alpha+\beta &= \frac{\alpha^2-\beta^2}{\alpha-\beta} = F_2 \\ \alpha^2+\alpha\beta+\beta^2 &= \frac{\alpha^3-\beta^3}{\alpha-\beta} = F_3 \\ \alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3 &= \frac{\alpha^4-\beta^4}{\alpha-\beta} = F_4 \\ &\cdots \end{aligned} \tag{6.1}$$

so that the n^{th} row sums to

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The *Fibonacci-lambda polynomials* $\Phi_m(x)$ will then have the recurrence relations

$$\Phi_{m+2}(x) = (\alpha + \beta x)\Phi_{m+1}(x) - \alpha\beta x\Phi_m(x), \quad m \geq 0, \tag{6.2}$$

$$\Phi_{m+1}(x) = \alpha\Phi_m(x) + (\beta x)^m, \quad m \geq 0. \tag{6.3}$$

The first few examples are

$$\begin{aligned}
&\Phi_0(x) = 0 \\
&\Phi_1(x) = 1 \\
&\Phi_2(x) = \alpha - \beta x \\
&\Phi_3(x) = \alpha^2 + \alpha\beta x + \beta^2 x^2 \\
&\Phi_4(x) = \alpha^3 + \alpha^2\beta x + \alpha\beta^2 x^2 + \beta^3 x^3 \\
&\Phi_5(x) = \alpha^4 + \alpha^3\beta x + \alpha^2\beta^2 x^2 + \alpha\beta^3 x^3 + \beta^4 x^4 \\
&\cdots
\end{aligned} \tag{6.4}$$

Clearly,

$$\Phi_m(1) = F_m.$$

6.2 "Fibonacci-Lucas Triangle"

To continue the Fibonacci theme in this section, we next form the triangle with elements $b_{i,j}$ (where i refers to rows and j to columns) defined by

$$b_{i,j} = b_{i-1,j} + b_{i-1,j-1}, \quad i \geq 2, \ 0 < j < i, \tag{6.5}$$

with boundary conditions

$$b_{i,0} = F_{i+2}, \ i \geq 0; \quad b_{i,i} = L_{i+1}, \ i \geq 1; \quad b_{i,j} = 0, j > i, \tag{6.6}$$

in which $L_n = \alpha^n + \beta^n$ represents the Lucas numbers. This yields the formation in Table 8. Note that (6.5) and (6.6) lead to $b_{i,1} = F_{i+3} = b_{i+1,0}$, $i \geq 1$.

TABLE 8. "Fibonacci-Lucas Triangle"

1							
2	3						
3	5	4					
5	8	9	7				
8	13	17	16	11			
13	21	30	33	27	18		
21	34	51	63	60	45	29	
34	55	85	114	123	105	74	47

This is termed a "Fibonacci-Lucas triangle" to distinguish it from the Fibonacci and Lucas triangles already n the literature [7]. The vertical and sloping sides of this triangle clearly have Fibonacci and Lucas numbers as their elements, but there are other connections, too.

6.3 Difference Operators

Instead of considering sums along rows, diagonals, and columns, we here look at differences between rows and columns by means of the row and column difference operators defined by

$$\Delta_r b_{i,j} = b_{i+1,j} - b_{i,j}, \tag{6.7}$$

$$\Delta_c b_{i,j} = b_{i,j+1} - b_{i,j}. \tag{6.8}$$

For example,

$$\begin{aligned}
\Delta_r b_{i,0} &= b_{i+1,0} - b_{i,0} && \text{by (6.7)}\\
&= F_{i+3} - F_{i+2} && \text{by (6.6)}\\
&= F_{i+1} &&\\
&= b_{i-1,0} && \text{by (6.6)}\\
&= b_{i,1} - b_{i,0} && \text{by (6.5) and } b_{i-1,1} = b_{i,0}\\
&= \Delta_c b_{i,0} && \text{by (6.8).}
\end{aligned}$$

More generally, Δ_r, Δ_c are commutative operations:

$$\begin{aligned}
\Delta_r \Delta_c b_{i,j} &= \Delta_r (b_{i,j+1} - b_{i,j}) && \text{by (6.8)}\\
&= (b_{i+1,j+1} - b_{i,j+1}) = (b_{i+1,j} - b_{i,j}) && \text{by (6.7)}\\
&= (b_{i+1,j+1} - b_{i+1,j}) - (b_{i,j+1} - b_{i,j}) &&\\
&= \Delta_c b_{i+1,j} - \Delta_c b_{i,j} && \text{by (6.8)}\\
&= \Delta_c \Delta_r b_{i,j} && \text{by (6.7).}
\end{aligned}$$

Other results can be investigated. For instance,

$$\Delta_r^j b_{i,j} = F_{i+2}. \tag{6.9}$$

We can prove (6.9) by means of mathematical induction on i and j.

By reversing the columns in Table 8 (that is, by making the Lucas numbers the left-hand exterior sloping side), one can also study these and other properties for a "Lucas-Fibonacci triangle"; this is a topic for further research. Are there, one might ask, any interesting relationships between the "Fibonacci-Lucas" and the "Lucas-Fibonacci" triangles?

7. CONCLUSION

7.1 Binary Extensions

These lambda-type triangles can be extended indefinitely. For instance, we can construct a triangle of binary numbers as in Table 9.

TABLE 9. Binary Triangle

1				
10	11			
100	101	111		
1000	1001	1011	1111	
10000	10001	10011	10111	11111

7.2 Determinants

Two other properties which are commonly examined are the values of corresponding determinants and modular arrays. The original left- and right-triangular lambda matrices (in Tables 2 and 3) have determinants with values which are powers of 3 and 2, respectively.

7.3 Modular Triangles

The displays in Tables 10 and 11 represent the original extended lambda triangle (Table 1) modulo 5 and modulo 7, respectively. Table 10 has symmetry in its odd rows and Table 11 has neat patterns of cycles. Further research could involve seeking a modulus which could produce remainders to develop specific patterns such as Sirpinski triangles [9], arrowhead curves [7], or the partitioning of the triangles into square arrays [3].

TABLE 10. Lambda Triangle Modulo 5

					1					
				2		3				
			4		1		4			
		3		2		3		2		
	1		4		1		4		1	
2		3		2		3		2		3

TABLE 11. Lambda Triangle Modulo 7

					1					
				2		3				
			4		6		2			
		1		5		4		6		
	2		3		1		5		4	
4		6		2		3		1		5

7.4 Ongoing Research

The purpose of this paper has been to explore some of the properties associated with the lambda triangle. In doing so, several ideas for further research have been suggested for the interested reader. Finally, in this spirit, one might extend the previous knowledge through negative numbers, that is, start with $-2, -4, -8, \ldots$ and $-3, -9, -27, \ldots$ (as in Table 1 with common vertex 1). All this has no physical or artistic relation to our original Timaeus panel. Indeed it is a world away from Plato and Michelangelo.

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REPORT ON THE TENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

The Tenth International Conference on Fibonacci Numbers and Their Applications held at Northern Arizona University in Flagstaff, Arizona, from June 24-28, 2002, found over 70 enthusiastic Fibonacci number lovers from Australia, Canada, England, Germany, Italy, Japan, Mexico, New Zealand, Poland, Romania, Scotland, and the USA gathered together to hear over 50 excellent presentations. The gathering was attended by both old and new Fibonacci friends, but it was sadly noted that several regulars were unable to be with us this year. They were both warmly remembered and greatly missed. A special thanks to organizer Cal Long and all the folks at Northern Arizona University for their hospitality and generosity in hosting this outstanding conference.

Monday through Wednesday morning found us savoring a variety of talks on things theoretical, operational, and applicable of a Fibonacci and related nature, with members sharing ideas while renewing old friendships and forming new ones.

Later on Wednesday the group was doubly treated. After the morning talks, we were entertained by mathemagician Art Benjamin's most impressive presentation; displaying his skills and cleverness by mentally performing challenging mathematical manipulations and zapping out magic squares as if (yes!) by magic. After graciously sharing some of the secrets of his wizardry with us, he dazzled one and all by mentally and accurately multiplying two five-place numbers to terminate his mesmerizing performance.

That afternoon we were bussed to our second wonder of the day: The Grand Canyon. Here we were able to spend several hours gazing at nature's wondrous spectacle. Oh to be a condor for an hour! In the evening a steak dinner was catered for us as we exchanged social and mathematical dialog to the background of exquisite scenic wonder at the edge of the Canyon. On the way back to the campus, we were able to witness a magnificent display of stars but an arm length away in the clear Arizona night sky.

On Thursday and Friday it was back to many more interesting, informative presentations and during the breaks we were treated to Peter Anderson's marvelous computer display of the many photographs he took of association members and their families enjoying the Canyon.

The closing banquet on Friday night terminated with a special tribute to Calvin T. Long for his very distinguished career of 50 years as teacher, mentor, and researcher, as well as valued friend, contributor to, and supporter of The Fibonacci Association. He was both praised and roasted by President Fred T. Howard and former editor Gerald E. Bergum. After much laughter and tears, Cal received a standing ovation from this proud and grateful group of his friends and colleagues.

After over an hour of cordial good-byes, everyone eventually drifted away vowing that, Lord willing, we'll all meet again in Braunschweig, Germany, in 2004.

Charles K. Cook