INEQUALITIES AMONG RELATED PAIRS OF FIBONACCI NUMBERS

K. T. Atanassov
CLBME-Bulgarian Academy of Sciences, PO Box 12, Sofia-1113, Bulgaria

Ron Knott
RDK Internet Consultancy, 1 Upperton Road, Guildford, GU2 7NB, UK

Kiyota Ozeki
Utsunomiya University, 350 Mine-machi, Utsunomiya City, Japan

A. G. Shannon
KvB Institute of Technology, North Sydney, 2060, and
Warrane College, The University of New South Wales, 2033, Australia

László Szalay
University of West Hungary, H-9400, Sopron, Bajcsy Zs, út 4, Hungary

(*Submitted November 2000-Final Revision January 2001*)

1. INTRODUCTION

What may be called "Fibonacci inequalities" have been studied in a variety of contexts. These include metric spaces [6], diophantine approximation [7], fractional bounds [5], Fibonacci numbers of graphs [1], and Farey-Fibonacci fractions [2]. Here we consider Fibonacci inequalities. The results are the "best possible" and we relate them through the sequence $\{m_r\}_{r=0}^{n}$ defined by

$$m_r = F_r F_{n+1-r},$$

where n is a fixed natural number and $F_1, F_2, F_3, \ldots$, are the ordinary Fibonacci numbers.

2. MAIN RESULT

Theorem: For every natural number k, the following inequalities for the elements of the sequence $\{m_k\}_{k=0}^{n}$ are valid:

(a) if $n = 4k$, then

$$F_1F_{4k} > F_3F_{4k-2} > \cdots > F_{2k-1}F_{2k+2} > F_{2k}F_{2k+1} > F_{2k-2}F_{2k+3} > \cdots > F_2F_{4k-1};$$

(b) if $n = 4k+1$, then

$$F_1F_{4k+1} > F_3F_{4k-1} > \cdots > F_{2k-1}F_{2k+3} > F_{2k+1}F_{2k+1} > F_{2k}F_{2k+2} > \cdots > F_2F_{4k};$$

(c) if $n = 4k+2$, then

$$F_1F_{4k+2} > F_3F_{4k} > \cdots > F_{2k+1}F_{2k+2} > F_{2k}F_{2k+3} > F_{2k-2}F_{2k+5} > \cdots > F_2F_{4k+1};$$

(d) if $n = 4k+3$, then

$$F_1F_{4k+3} > F_3F_{4k+1} > \cdots > F_{2k+1}F_{2k+3} > F_{2k+2}F_{2k+2} > F_{2k}F_{2k+4} > \cdots > F_2F_{4k+2}.$$

Examples when $k = 3$:

(a) $F_1F_{12} = 144 > F_3F_{10} = 110 > F_5F_8 = 105 > F_6F_7 = 104 > F_4F_9 = 102 > F_2F_{11} = 89;$

(*b*) $F_1F_{13}=233>F_3F_{11}=178>F_5F_9=170>F_7F_7=169>F_6F_8=168$
$>F_4F_{10}=165>F_2F_{12}=144;$

(*c*) $F_1F_{14}=377>F_3F_{12}=288>F_7F_8=273>F_6F_9=272>F_4F_{11}=267>F_2F_{13}=233;$

(*d*) $F_1F_{15}=610>F_3F_{13}=466>F_7F_9=442>F_8F_8=441>F_6F_{10}=440$
$>F_4F_{12}=432>F_2F_{14}=377.$

Proof of Theorem: We shall use induction simultaneously on n, that is, on k and i.
For $k=1$, we have

$$\begin{aligned}
&F_1F_4=1\times 3>2\times 1=F_3F_2,\\
&F_1F_5=1\times 5>2\times 2=F_3F_3>1\times 3=F_2F_4,\\
&F_1F_6=1\times 8>2\times 3=F_3F_4>1\times 5=F_2F_5,\\
&F_1F_7=1\times 13>2\times 5=F_3F_5>3\times 3=F_4F_4>1\times 8=F_2F_5.
\end{aligned}$$

Assume that inequalities (a), (b), (c), and (d) are true for some $k\geq 1$. Then we must prove that the inequalities are true for $k+1$.

In particular, from the truth of (d), it follows that, for every i, $1\leq i\leq k$:

$$F_{2i+1}F_{4k-2i+3}>F_{2i+2}F_{4k-2i+2}. \tag{2.1}$$

We shall discuss case (a), but the other cases are proved similarly.

First, we see that

$$F_1F_{4k+4}-F_3F_{4k+2}=F_{4k+3}+F_{4k+2}-2F_{4k+2}=F_{4k+3}-F_{4k+2}>0,$$

that is, the inequality

$$F_{2i-1}F_{4k-2i+6}>F_{2i+1}F_{4k-2i+4} \tag{2.2}$$

is valid for $i=1$.

Let us assume that, for some i, $1\leq i\leq k$, inequality (2.2) is true. Then we must prove that the inequality

$$F_{2i+1}F_{4k-2i+4}>F_{2i+3}F_{4k-2i+2} \tag{2.3}$$

is also true.

But

$$\begin{aligned}
F_{2i+1}F_{4k-2i+4}-F_{2i+3}F_{4k-2i+2}&=F_{2i+1}F_{4k-2i+3}+F_{2i+1}F_{4k-2i+2}-F_{2i+1}F_{4k-2i+2}-F_{2i+2}F_{4k-2i+2}\\
&=F_{2i+1}F_{4k-2i+3}-F_{2i+2}F_{4k-2i+2}>0
\end{aligned}$$

because of the inductive assumption for (d) in case (2.1). Therefore, inequality (3) holds, from which the truth of (a) follows.

3. DISCUSSION

By analogy with the extremal problems discussed in [3], we can formulate the following.

Corollary: For every natural number n the maximal element of the sequence $\{m_k\}_{k=0}^{n}$ is F_1F_n.

Proof of Corollary: Equation (I_{26}) of [8] can be rewritten as

$$F_n=F_{k+1}F_{n-k}+F_kF_{n-k-1}$$

to show that $F_n>F_{n-k}F_{k+1}$, $1\leq k\leq n$, which gives the required result.

More generally, F_1F_n is the maximal element of the set

$$M = \{F_{i_1}F_{i_2}, \ldots, F_{i_{k-1}}F_{i_k}\},$$

where $n = i_1 + i_2 + \cdots + i_k$, $1 < k \leq n$, is a positive partition into k parts. The result can be proved by induction.

4. CONCLUDING COMMENTS

A somewhat analogous result was proposed by Bakinova [4] and proved by Mascioni [9]:

$$\frac{F_{k+1}}{F_1} < \frac{F_{k+3}}{F_3} < \frac{F_{k+5}}{F_5} < \cdots < \alpha^k < \cdots < \frac{F_{k+6}}{F_6} < \frac{F_{k+4}}{F_4} < \frac{F_{k+2}}{F_2}.$$

The results can be generalized to the sequence $\{u_n(0, 1; p, q)\}$ defined by the recurrence

$$u_n = pu_{n-1} - qu_{n-2}, \; n \geq 2,$$

with $u_0 = 0$, $u_1 = 1$, $p \neq 0$, $p \in \mathbb{Z}$, $\Delta = p^2 + 4q > 0$, in which case a neater proof comes from the use of the Binet forms for the general terms and hyperbolic cosines and sines. The result can also be extended to related triples of Fibonacci numbers.

ACKNOWLEDGMENT

Appreciation is expressed for comments by an anonymous referee and for references suggested by the editor.

REFERENCES

1. A. F. Alameddine. "Bounds on the Fibonacci Number of a Maximal Outerplanar Graph." *The Fibonacci Quarterly* **36.3** (1998):206-10.
2. K. Alladi & A. G. Shannon. "On a Property of Farey-Fibonacci Fractions." *The Fibonacci Quarterly* **15.2** (1977):153-55.
3. K. T. Atanassov. "One Extremal Problem." *Bulletin of Number Theory & Related Topics* **8.3** (1984):6-12.
4. V. Bakinova. "Problem B-486." *The Fibonacci Quarterly* **20.4** (1982):366.
5. J. H. E. Cohn. "Recurrent Sequences Including *N*." *The Fibonacci Quarterly* **29.1** (1991): 30-36.
6. M. Deza & M. Laurent. "The Fibonacci and Parachute Inequalities for l_1-metrics." *The Fibonacci Quarterly* **30.1** (1992):54-61.
7. C. Elsner. "On Diophantine Approximations with Rationals Restricted by Arithmetical Conditions." *The Fibonacci Quarterly* **38.1** (2000):25-34.
8. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*, p. 59. Boston: Houghton-Mifflin, 1969.
9. V. D. Mascioni. "Solution to Problem B-486." *The Fibonacci Quarterly* **21.4** (1983):309-10.

AMS Classification Number: 11B39

❖❖❖