SOME FRACTALS IN GOLDPOINT GEOMETRY

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Dedicated to the memory of Herta T. Freitag

1. INTRODUCTION

Problems of interest in goldpoint[1] geometry [1] arise from study of tile-figures that are obtained when goldpoints are marked on sides of triangles, squares, pentagons, etc. and joined by lines in various ways. Many combinatoric problems arise naturally in the course of such studies. Another type of problem is to determine how to combine collections of golden tiles in jig-saw fashion, so that they tile a given geometric figure (or the whole plane) with goldpoint marks on touching sides corresponding everywhere.

Examples of these types of problems are the following:

(i) Find how many different golden tiles can be formed from regular polygons; that is, find how many inequivalent golden triangles, squares, pentagons, etc. there are.

(ii) Given a regular hexagon, find how many different ways it can be tiled by equilateral golden triangles, jig-saw fashion.

In this paper I introduce a new type of problem into goldpoint geometry. I study a variety of fractals which are achieved by using as base the segment [0, 1], and a motif which involves the goldpoints of that segment.[2]

In Sections 2 and 3, the goldpoint dust set and snowflake are defined, and some of their properties are derived.

In the following section, I describe goldpoint fractals which I dedicate to the memory of the inspirational American mathematician Herta T. Freitag, who passed away early in 2000 in her 91st year.

In the final section, I present studies of fractals which are based on the regular pentagon. It is well-known (indeed the knowledge goes back to extreme antiquity, since it is mentioned in cabalistic literature) that the golden mean occurs frequently in the geometry of the pentagon [3] and its accompanying pentagram star. It is hoped that the results given below on pentagon fractals will add to existing literature on the pentagram.

2. THE GOLDPOINT DUST SET

We define the goldpoint dust set (the *gp-dust set*) by prescribing an infinite process similar to that used to produce Cantor's fractal set.

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[1] A point P in a segment AB is a *goldpoint* of AB if AP/PB is either α or $1/\alpha$ (α is the golden ratio).

[2] The terms 'base' and 'motif' are now well known. Excellent references for these terms, and for several of the analytic techniques used in this paper are [2] and [4].

We take the unit line-segment [0, 1] on the x-axis, and compute its goldpoints, which are at points $(\alpha^{-1}, 0)$ and $(\alpha^{-2}, 0)$; call these points G_1 and G_2, respectively. Then we discard all points in the open set of the segment (G_1G_2).

Next we compute the positions of the goldpoints H_1, H_2 and H_3, H_4 of the two remaining segments $[G_1, 1]$ and $[0, G_2]$, respectively. Then we discard the two open sets between these two pairs of goldpoints.

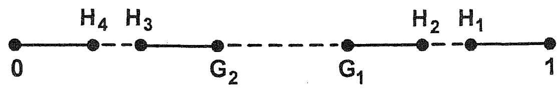


FIGURE 1. Stage 2 of the formation of the goldpoint dust set

We continue this process *ad infinitum*, at each stage discarding all the central open sets between pairs of goldpoints.

The limiting set of points is called the goldpoint dust set of [0, 1]. All points in it, except the two endpoints, are goldpoints of some segment in [0, 1].

Some properties of points in the gp-dust set are described next.

G_1, G_2 are the goldpoints of line segment [0, 1], and G_3, G_4 are the goldpoints of $[0G_2]$.

Measuring lengths from 0, and writing G_i for $\|[0G_i]\|$, we find:

$$
\begin{aligned}
G_1 &= 1/\alpha^2 + 1/\alpha^3 = 1/\alpha &&= F_{-1}\alpha + F_{-2}\\
G_2 &= 1/\alpha^2 = -\alpha + 2 &&= F_{-2}\alpha + F_{-3}\\
G_3 &= 1/\alpha^4 + 1/\alpha^5 = 1/\alpha^3 &&= F_{-3}\alpha + F_{-4}\\
G_4 &= 1/\alpha^4 &&= F_{-4}\alpha + F_{-5}
\end{aligned}
$$

and so on.

Similarly, H_1, H_2 are the goldpoints of line segment $[G_1, 1]$, and for them we find:

$$
\begin{aligned}
H_1 &= 1/\alpha + 1/\alpha^4 + 1/\alpha^5 = 1/\alpha + 1/\alpha^3\\
H_2 &= 1/\alpha + 1/\alpha^4
\end{aligned}
$$

It may be noted that:

G_1 is a goldpoint of [0, 1] (given),
G_1 is a goldpoint of $[G_2H_2]$ (since $G_2G_1 = \alpha^{-3}$ and $G_1H_2 = \alpha^{-4}$),
G_1 is a goldpoint of $[G_3H_1]$ (since $G_3G_1 = \alpha^{-2}$ and $G_1H_1 = \alpha^{-3}$).

It follows that, as the process of discarding central open segments continues, all of the points left in the dust set are goldpoints (0 and 1 are excluded); in the limit, each point is a goldpoint an infinite number of times, with respect to pairs of other points in the dust set. It might be appropriate to call this the gold-dust set.

It is evident from the above analysis that each goldpoint in the dust set can be expressed uniquely in α-nary form thus:

$$\text{goldpoint} = 0.c_1c_2c_3\cdots \equiv c_1\alpha^{-1} + c_2\alpha^{-2} + c_3\alpha^{-3} + \cdots,$$

where all the c_i coefficients are zero or unity, and with no pair of adjacent coefficients being $(1,1)$.*

* If in the calculation of a goldpoint we obtain both $c_i = 1$ and $c_{i+1} = 1$, we are required to combine the adjacent terms, using $\alpha^{-i} + \alpha^{-(i+1)} = \alpha^{-(i-1)}$.

Examples:

$$G_1 = 0.1,\ G_2 = 0.01,\ G_3 = 0.001,\ \text{etc.}$$
$$H_1 = G_1 + \alpha^{-4} + \alpha^{-5} = G_1 + \alpha^{-3} = 0.101,$$
$$H_2 = G_1 + \alpha^{-4} = 0.1001.$$

The goldpoint dust set is the set of all points in (0, 1) which have this type of α-ternary form (reminiscent of maximal Zeckendorf representations of n in terms of the Fibonacci numbers).

3. THE GOLDPOINT SNOWFLAKE

The following diagrams show how a snowflake fractal (*à la* von Koch, 1904) can be constructed from a line segment base, and the motif given as phase 1 in Figure 2. G and H are the goldpoints of the line segment [0, 1]. Phases 2 and 5 indicate how the fractal develops. Since $OG = H1 = 1/\alpha^2$, and the reduction factor is $r = \alpha^2$ at each step, the length of the perimeter of the snowflake at phase p is $P_p = (4/\alpha^2)^p$ for $p = 0, 1, 2, \ldots$.

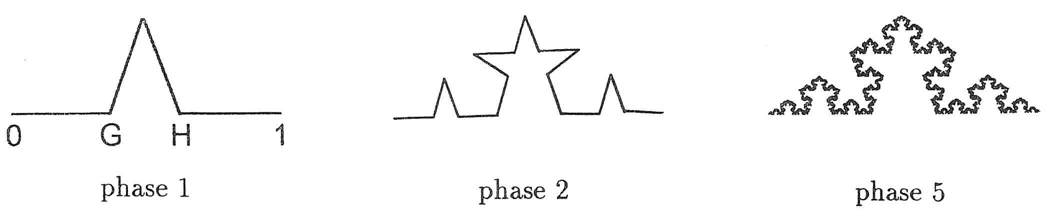


FIGURE 2. Development phases of the goldpoint snowflake

Fractal (or self-similarity) dimension

At each step, from each segment $m = 4$ new segments are formed, with length reduction factor $r = \alpha^2$ in every case. Hence, the fractal dimension of the goldpoint snowflake is

$$d = \frac{\log m}{\log r} = \frac{\log 4}{2\log\alpha} = 1.44042\ldots.$$

4. HERTA'S SHIELD, STAR JEWEL AND COMB

In the last few months of Herta Freitag's life, I sent her three goldpoint fractal diagrams, which I hoped would amuse her. The shield (I said) was for her protection, and was drawn on her 90th birthday card. The jewel for her dress and the comb for her hair were sent later with get-well messages. Sadly, my shield did not avail her for long; however, I was sure that she would appreciate the diagrams and look for the relationships to the golden mean that are evident within them.

Both the shield and the star jewel are developed with goldpoint snowflakes on the sides of an equilateral triangle. The shield is exterior to the triangle; the jewel is interior to it (see Figs. 3 and 4).

Figure 5 shows Herta's goldpoint comb; I imagined it to be made of ivory. In the limit, it has an infinite number of teeth, the prong points forming a set of Hausdorff measure zero and equivalent to the gp-dust set. I don't know what it would have done to her hair. It is easy to see how the comb is built up of rectangles erected upon line segments parallel to those 'left in' during the process of obtaining the goldpoint set (see Fig. 2). Upon each segment, a golden rectangle is constructed, with the horizontal segment being the larger side.

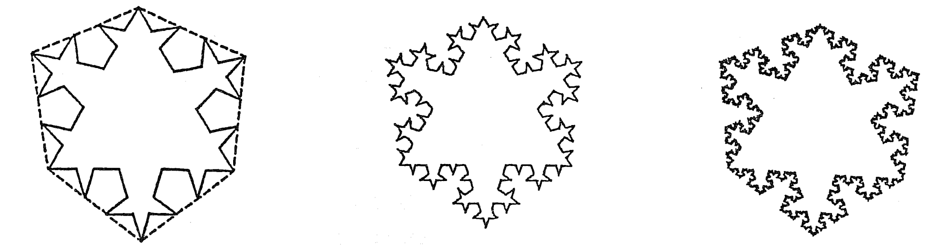


FIGURE 3. Herta's goldpoint shield (phases 2, 3, and 5)
[The dotted bounding-polygon is added in 2 to demonstrate the shield's outer shape.]

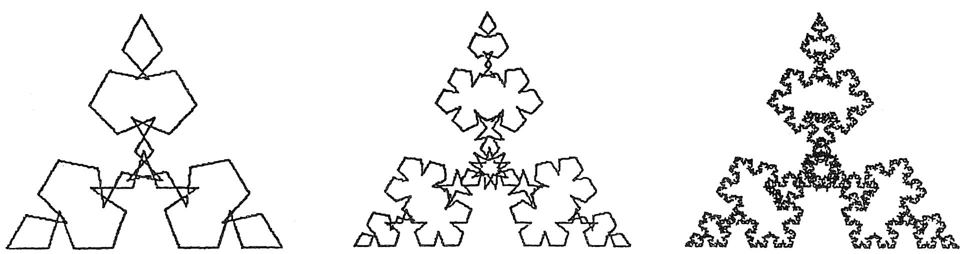


FIGURE 4. Herta's star jewel (phases 2, 3, and 5)

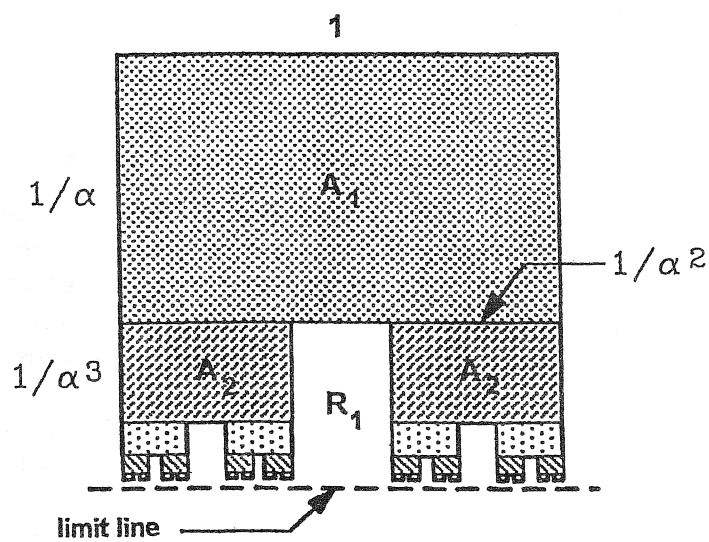


FIGURE 5. Herta's goldpoint comb

Figure 5 shows how the short sides of the rectangles have lengths in the sequence:

$$\frac{1}{\alpha}, \frac{1}{\alpha^3}, \frac{1}{\alpha^5}, \frac{1}{\alpha^7}, \ldots.$$

This is a geometric progression of common ratio $1/\alpha^2$, and its infinite sum is 1. Therefore, the goldpoint comb has height 1 and it covers (in the limit, and except for the limit line) a square of side 1.

Thus, the unit square of the comb is tiled by golden rectangles in an interesting way.

If we check the 'hole' or 'spaces' in the comb, we see that they are also rectangles, all standing on the horizontal limit line where the teeth 'end'. Again checking the dimensions, we see that each of these rectangles is also a golden rectangle. Moreover, the largest 'hole' rectangle is equal to the second largest ivory rectangle; the second largest 'hole' rectangle is equal to the third largest ivory rectangle; and so on.

The area (A) of the ivory, and the area (H) of the 'holes'

Working directly from Figure 5 we get, for the total ivory in the comb:

$$A = 1\times\frac{1}{\alpha}+2\times\frac{1}{\alpha^5}+4\times\frac{1}{\alpha^9}+8\times\frac{1}{\alpha^{13}}+\cdots=\frac{1}{\alpha}\sum_{i=1}^{\infty}\left(\frac{2}{\alpha^4}\right)^i=\frac{\alpha^2}{3}.$$

Then, for the area of the 'holes' in the comb:

$$H = 1-A = 1-\frac{1}{3}\alpha^2=\frac{1}{3\alpha^2}.\ \text{[Check: } (\alpha^2+\alpha^{-2})=3.\text{]}$$

5. THE GOLDPOINT MOTIF TRIANGLE, AND PENTAGON FRACTALS

In this final section we first analyze the goldpoint motif triangle, showing various ways by which it can be partitioned.

Then we take a regular pentagon and study some of its goldpoint properties. We show how a fractal of pentagon fractals can be constructed within it, and point out one or two of the properties of this object.

Properties of the goldpoint motif bounding triangle

In Figure 6(a) below, the goldpoint motif *AGCHB* is shown, together with its bounding triangle *ABC*. (It was also shown in Fig. 2 above.) This triangle partitions into two (108°, 36°, 36°) triangles, viz. *AGC* and *BHC*, which we call *S*-triangles, and a (36°, 72°, 72°) triangle, *GHC*, which we call a *T*-triangle. We shall use the convention S_i to describe an *S*-triangle drawn on a base line segment of length $1/\alpha^i$, $i = 0, 1, 2, \ldots$; similarly, we shall use T_i for the *T*-triangles drawn on such base line segments.

When making the analyses and calculations, we shall have recourse to the formulas given at the beginning of Section 2, and also to the following trigonometric relations:

θ	36°	72°
$\sin\theta$	$\sqrt{\alpha+2}/(2\alpha)$	$(1/2)\sqrt{\alpha+2}$
$\cos\theta$	$\alpha/2$	$1/(2\alpha)$
$\tan\theta$	$\sqrt{\alpha+2}/\alpha^2$	$\alpha\sqrt{\alpha+2}$

The goldpoint motif triangle, and some partitions of it

Figure 6(a) is used to demonstrate several partition properties of the goldpoint motif triangle. Figure 6(b) shows how the triangle can be partitioned by pentagrams and S-triangles of diminishing sizes and with sides $1/\alpha^i$. Various calculations and comments on these figures are given below the diagrams.

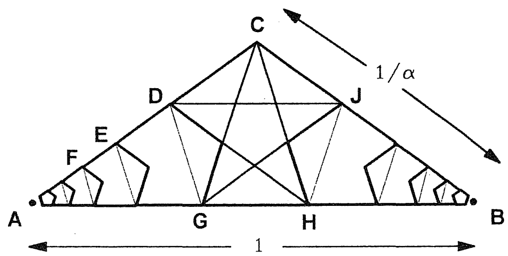


FIGURE 6(a). The motif triangle and some dividing lines

If $AB = 1$, then it is immediately seen that ABC is an S_0 triangle, which is partitioned by GC and HC into two S_1 and one T_3 triangles (since $AC = BC = 1/\alpha$ and $GH = 1/\alpha^3$). Thus, $S_0 = 2S_1 \cup T_3$.

The area of triangle ABC is $(1/2)AC \sin 36° = \sqrt{\alpha+2}/(4\alpha^2)$.

Other partitions of ABC can be seen in the constructions. For example, the two T_3 triangles ADG and BJH together with the central pentagon P_3 on GH. Another is the set of decreasing and overlapping pentagons, on sides CD, DE, EF, ... and, similarly, on the right side of center, whose union limitingly fills triangle ABC.

Finally, we observe that since an S-triangle can be partitioned into a T-triangle and an S-triangle (e.g., $ABC = AGC \cup GCB$), by repeated divisions ABC can be partitioned into a sequence of diminishing S-triangles; or else, similarly, into a sequence of diminishing T-triangles. We won't spell out their relative sizes, but point out that they are all in ratios of powers of α.

Figure 6(b) demonstrates how the golden motif triangle can be partitioned into an attractive double sequence of diminishing pentagrams, with sides in diminishing powers of α, together with sequences of diminishing S-triangles.

Proposition: Every pentagram vertex (except C) is a double goldpoint with respect to two pairs of pentagram vertices.

Proof: By inspection of the largest pair of pentagrams, and induction.

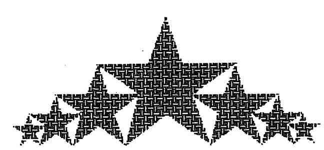


FIGURE 6(b). Pentagrams and S-triangles constructed in the motif triangle

The complement in ΔABC of the infinite set of (interiors of) pentagrams is an infinite set S of S-triangles, being $3S_3 \cup 8S_4 \cup 8S_5 \cup \cdots$. This can be regarded as phase 1 of a fractal. In the next phase, every S-triangle in phase 1 provides a similar figure, all S-triangles in it being reduced by α^{-3}.

The dust set of this fractal is the set of all vertices of S-triangles produced in this multiply-infinite recurrence process.

Some properties of the regular pentagon, with goldpoints and partitions

The next two figures, 6(c) and 6(d), show regular pentagons, of side 1, with various construction lines upon them.

In Figure 6(c), ΔABC is a T_1-triangle, so $AC = 1/(2\cos 72) = \alpha$. From ΔAGD, we get $AG = 1/(2\cos 54) = \alpha/\sqrt{\alpha+2}$ and $GD = (1/2)\tan 54 = \alpha^2/2\sqrt{\alpha+2}$. Also, $CD = (1/2)\tan 72 = (1/2)\alpha\sqrt{\alpha+2}$.

By similar pentagons, $G'D = \alpha^{-3}GD$ and $G''A = \alpha G'/\sin 54 = 2/\sqrt{\alpha+2}$.

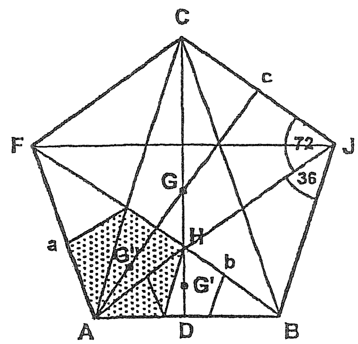


FIGURE 6(c). A regular pentagon

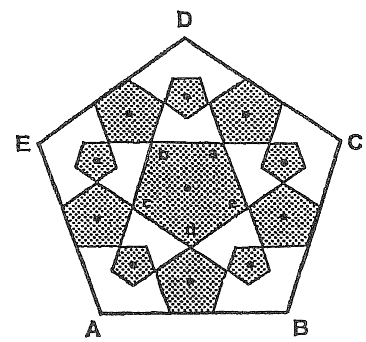


FIGURE 6(d). A fractal of pentagons

Proposition:

(i) $GG' = GG''$.

(ii) G is a goldpoint of CG'.

Proof:

(i)

$$\begin{aligned} GG' &= GD - G'D \\ &= \alpha^2/2\sqrt{\alpha+2} - 1/(2\alpha\sqrt{\alpha+2}) \\ &= 1/\sqrt{\alpha+2} \text{ (since } \alpha^2 - 1/\alpha = 2) \end{aligned}$$

and

$$\begin{aligned} GG'' &= GA - G''A \\ &= \alpha/\sqrt{\alpha+2} - 1/(\alpha\sqrt{\alpha+2}) \\ &= 1/\sqrt{\alpha+2} \text{ (since } \alpha - 1/\alpha = 1). \end{aligned}$$

Therefore

$$GG' = GG''.$$

(ii)

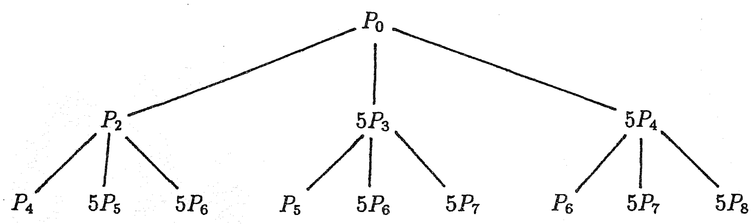
$$\begin{aligned} GG'/GC &= GG'/GA \\ &= (1/\sqrt{\alpha+2})\cdot(\sqrt{\alpha+2})/\alpha = 1/\alpha. \end{aligned}$$

Other results about goldpoints in a pentagon construction may be found on page 28 in [2]. Let us turn to Figure 6(d) and examine the fractal of pentagons.

It is evident how Figure 6(d) can be obtained from Figure 6(c). The shaded pentagon P_2 is replaced by its inner pentagon (a P_4), and then the two small pentagons are replicated around pentagon *abcde* (a P_2).

Looking at Figure 6(d), we see $5P_3$'s and $5P_4$'s arranged alternately with their centers on a circle by Proposition (i) above, and with a pentagon P_2 in the middle. We can regard this as a motif for constructing a fractal of pentagons in the interior of pentagon *ABCDE*.

Thus, to arrive at phase 1, we must remove all points in the unshaded regions, together with the perimeter of *ABCDE*. Then, to arrive at phase 2, we repeat the above constructions and removals in each of the eleven shaded pentagons. What remains will be 121 shaded pentagons, each scaled by a factor of α^i, $i = 2, 3$, or 4 according to its construction. From the tree diagram below, we see that the distribution of pentagons will then be $1P_4$, $10P_5$, $35P_6$, $50P_7$, $25P_8$.



Evidently, this process can be continued indefinitely. And formulas can be computed for the coefficients on the tree and for reduction factors in areas when passing from phase i to phase $i+1$.

The dust set of the fractal is the set of points in *ABCDE* which are not removed by this infinite process. A moment's thought shows that this set consists of the centers of all the pentagons constructed in the 'whole' process. And the set consists of a *cosmos* of points arranged in circles, with similar, reduced, circles arranged around each of them, and so on *ad infinitum*. Because of the similarity of this system with Ptolomy's model of the Universe, we name this dust set the **Ptolomaic dust set**.

The next two figures show phases of the interior and exterior fractals which are constructed on a regular pentagon using the goldpoint motif on its sides.

Phase 2 of Figure 6(e) shows an attractive clover-leaf arrangement of five leaves, each of three P_3 pentagons, formed in P_2's and arranged around a central P_2.

Phase 4 shows clearly how the interior goldpoint fractal of a regular pentagon is equal to the exterior goldpoint fractal of its pentagram.

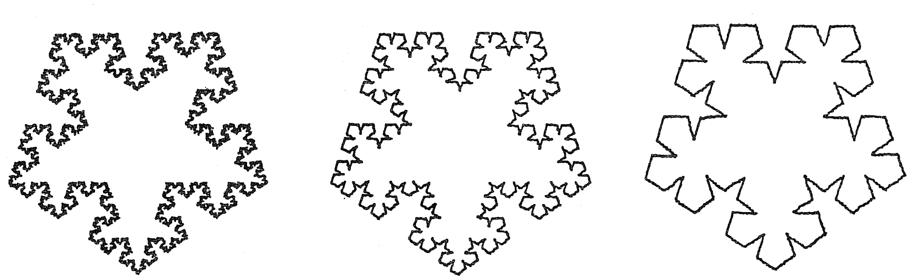


FIGURE 6(e). Interior goldpoint fractal of a pentagon (phases 2, 3, and 4)

It is clear from the phase 1 diagram of Figure 6(f) that the exterior goldpoint fractal of a regular pentagon is bounded by a regular pentagon. It is easy to prove this using angle values of the S- and T-triangles which touch the boundary. We believe this property of a von Koch-type fractal having a bounding polygon which is similar to the generating polygon to be unique.

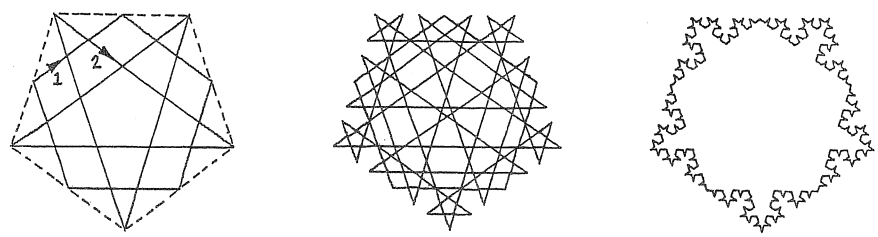


FIGURE 6(f). Exterior goldpoint fractal of a pentagon (phases 1, 2, and 3)

A final interesting comment is the following: the sharp boundary points in the phase 1 diagram can be connected by a unicursal polygon of chords of the diagram (each chord begins and ends along an arm of a point-angle), whereas the sharp boundary points of the phase 2 diagram require two such unicursal polygons to join them all up. In phase n, there will be 2^n unicursal polygons required. The unicursal perimeters can be calculated in terms of α, given that P_0 has side length 1. For example, in P_1, the unicursals have perimeters 5 and $5(7-3\alpha)$, respectively.

ACKNOWLEDGMENT

I wish to thank Dr. Hans Walser for his valuable comments on this paper. Dr. Walser is the author of a book [5] entitled *Der Goldene Schnitt* [The Golden Section], which contains a chapter that presents other fractals involving the golden ratio in their construction.

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