STERN'S DIATOMIC ARRAY APPLIED TO FIBONACCI REPRESENTATIONS

Marjorie Bicknell-Johnson

665 Fairlane Avenue, Santa Clara, CA 95051-5615

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1. STERN'S DIATOMIC ARRAY

Each row of Pascal's triangle is formed by addition of adjacent elements of the preceding row, producing binomial coefficients and counting combinations. Each row of Stern's diatomic array is formed by addition of adjacent elements of the preceding row, but interspersed with elements of the preceding row. In this case, the rows of the table will be shown to count certain Fibonacci representations.

Starting with 1 and 1, form a table in which each line is formed by copying the preceding line, and inserting the sum of consecutive elements: 1, 1; 1, 1+1, 1; 1, 1+2, 2, 2+1, 1; .... The body of the table contains Stern's diatomic array, sequence A049456 in [10]. Actually, this array has been called Stern's diatomic series in the literature [9], [11], but it should have been called the Eisenstein–Stern diatomic series by earlier authors because Stern's introduction refers back to Eisenstein.

1																																1
1																2																1
1								3								2								3								1
1				4				3				5				2				5				3				4				1
1		5		4		7		3		8		5		7		2		7		5		8		3		7		4		5		1
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11	3	10	7	11	4	9	5	6	1
...		...		...		...		...		...		...		...		...		...		...		...		...		...		...		...		...

If $a_{n,k}$ is the k^{th} term in the n^{th} line, $k = 1, 2, \ldots, n = 0, 1, 2, \ldots,$

$$a_{n,2m} = a_{n-1,m} + a_{n-1,m+1} \text{ and } a_{n,2m-1} = a_{n-1,m}. \tag{1.1}$$

Lehmer [9] reports that Stern took the initial line 1, 1 as the zeroth line and proved, among others, the following properties:

1. The number of terms in the n^{th} line is $2^n + 1$, and their sum is $3^n + 1$.
2. The table is symmetric; in the n^{th} line the k^{th} term equals the $(2^n + 2 - k)^{th}$ term.
3. Terms appearing in the n^{th} line as sums of their two adjacent terms are called dyads. There are 2^{n-1} dyads and $2^{n-1} + 1$ non-dyads on the n^{th} line. The dyads a occupy positions of even number k (called rank) on the line.
4. Two consecutive terms, a and b, have no common factor.
5. Every ordered pair (a, b) occurs exactly once as consecutive terms in some line of the table.
6. If a and b are relatively prime, the pair of consecutive terms (a, b) appears in the line whose number is one less than the sum of the quotients appearing in the expansion of a/b in a regular continued fraction.

Lehmer [9] then uses the quotients of the continued fraction expansion of r_1/r_2 to place the consecutive terms r_1 and r_2 into the table by computing both the line number and the rank of r_1. Further, he shows that the largest dyads in the n^{th} line, $n \geq 2$, have the value F_{n+2}, the $(n+2)^{nd}$ Fibonacci number. Lehmer's results for the line number and rank are summarized in Theorem 1.1.

Theorem 1.1: If consecutive terms r_1 and r_2 occur on the n^{th} line of Stern's diatomic array and if the continued fraction for r_1/r_2 is $[q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, then

$$n = q_1 + q_2 + q_3 + \cdots + q_{m-2} + r_{m-1} - 1, \tag{1.2}$$

and if m is odd (even), r_1 is the left (right) neighbor of r_2 in the first (second) half of line n. If m is odd, the position number k (rank) of r_1 in the first half of line n is

$$k = 2^{q_1+q_2+\cdots+q_{m-2}} - 2^{q_1+q_2+\cdots+q_{m-3}} + \cdots - 2^{q_1+q_2} + 2^{q_1}. \tag{1.3}$$

More recently, Calkin and Wilf [6] use Stern's diatomic array to explicitly describe a sequence $b(n)$ (sequence A002487 in [10]) such that every positive rational appears exactly once as $b(n)/b(n+1)$,

$$\{b(n)\} = \{1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, \ldots\}. \tag{1.4}$$

It is shown in [6] that $b(n)$ counts the number of hyperbinary representations of the integer $n, n \geq 1$; that is, the number of ways of writing n as a sum of powers of 2, each power being used at most twice, $b(0) = 1$. Here, we apply Stern's diatomic array to counting Fibonacci representations.

2. FIBONACCI REPRESENTATIONS

Let $R(N)$ denote the number of Fibonacci representations [4] of the positive integer N; that is, the number of representations of N as sums of distinct Fibonacci numbers F_k, (or as a single Fibonacci number F_k), $k \geq 2$, written in descending order. We define $R(0) = 1$. The Zeckendorf representation of N, denoted Zeck N, is the unique representation of N using only non-consecutive Fibonacci numbers F_k, $k \geq 2$. The largest Fibonacci number contained in N will be listed first in Zeck N. Whenever $R(N)$ is prime, Zeck N uses only Fibonacci numbers whose subscripts have the same parity [3], [5]. For that reason, we are interested in integers N whose Zeckendorf representation uses only even-subscripted Fibonacci numbers; we call such N an even-Zeck integer, denoted $\tilde{N}$, sequence A054204 in [10]. The j^{th} even-Zeck integer $\tilde{N} = \tilde{N}(j)$ can be written immediately when j is known.

We list early values $R(\tilde{N})$ for consecutive even-Zeck integers $\tilde{N}$, augmented with $R(0) = 1$, sequence A002487 in [10]:

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$\tilde{N}(j)$	0	1	3	4	8	9	11	12	21	22	24	25	29	30	32	33	...
$R(\tilde{N}(j))$	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	...

(2.1)

The even-Zeck integers $\tilde{N}(j)$ are enumerated below for $j = 1, 2, \ldots$; we define $\tilde{N}(0) = 0$.

j	binary	powers of 2	Zeck $\tilde{N}(j)$	$\tilde{N}(j)$
1	1	2^0	F_2	1
2	10	2^1	F_4	3
3	11	$2^1 + 2^0$	$F_4 + F_2$	4
4	100	2^2	F_6	8
5	101	$2^2 + 2^0$	$F_6 + F_2$	9
6	110	$2^2 + 2^1$	$F_6 + F_4$	11
7	111	$2^2 + 2^1 + 2^0$	$F_6 + F_4 + F_2$	12
8	1000	2^3	F_8	21
9	1001	$2^3 + 2^0$	$F_8 + F_2$	22
10	1010	$2^3 + 2^1$	$F_8 + F_4$	24
...	...	...	...	...

Lemma 2.1: If j is represented as the sum of distinct powers of 2 in descending order, $j = 2^r + 2^s + \cdots + 2^w, r > s > w$, then the j^{th} even-Zeck integer $\tilde{N} = \tilde{N}(j)$ is given by Zeck $\tilde{N}(j) = F_{2(r+1)} + F_{2(s+1)} + \cdots + F_{2(w+1)}$. In short, replace the binary representation of j in the powers $2^p, p = 0, 1, \ldots,$ by $F_{2(p+1)}$ to find $\tilde{N} = \tilde{N}(j)$.

Proof: The short table displays Lemma 2.1 for $j = 1, 2, \ldots, 10$. The next even-Zeck integer $\tilde{N}(j+1)$ will be formed from the binary representation of $(j+1)$. □

Lemma 2.2. (i) If Zeck $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, has F_2 for its smallest term, then $\tilde{N} - 1 = \tilde{N}(j-1)$, but $\tilde{N} + 1$ is not an even-Zeck integer.

(ii) If Zeck $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, has F_{2c}, $c \geq 2$, for its smallest term, then $\tilde{N} + 1 = \tilde{N}(j+1)$, but $\tilde{N} - 1$ is not an even-Zeck integer.

(iii) The even-Zeck integer $\tilde{N}^*$ preceding $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, with F_{2c}, $c \geq 1$, for its smallest term, is $\tilde{N}(j-1) = \tilde{N}^* = \tilde{N} - F_{2c-2} - 1$.

Proof: Let $c = 1$, and take $\tilde{N}(j) = F_{2n} + \cdots + F_{2p} + F_2$, $p \geq 2$, $n \geq 3$. Then $\tilde{N} - 1 = \tilde{N}(j-1)$, but $\tilde{N} + 1 = F_{2n} + \cdots + F_{2p} + F_3$ is not an even-Zeck integer, illustrating (i). Further, $\tilde{N}(j-1) = \tilde{N} - F_0 - 1 = \tilde{N} - F_{2c-2} - 1, c = 1$, satifying (iii).

Let $\tilde{N}(j) = F_{2n} + \cdots + F_{2c+2p} + F_{2c}$, $c \geq 2$, $p \geq 1$, $n \geq 3$. Then $\tilde{N}(j) + 1 = F_{2n} + \cdots + F_{2c+2} + F_{2c} + F_2 = \tilde{N}(j+1)$, but $\tilde{N}(j) - 1 = F_{2n} + \cdots + F_{2c+2p} + F_{2c} - 1 = F_{2n} + \cdots + F_{2c+2p} + (F_{2c-1} + \cdots + F_7 + F_5 + F_3)$, not an even-Zeck integer, as in (ii). Part (iii) follows from

$$\begin{aligned}\tilde{N}(j) - F_{2c-2} - 1 &= F_{2n} + \cdots + F_{2c+2p} + F_{2c} - F_{2c-2} = 1 \\ &= F_{2n} + \cdots + F_{2c+2} + (F_{2c-1} - 1) \\ &= F_{2n} + \cdots + F_{2c+2p} + (F_{2c-2} + \cdots + F_6 + F_4 + F_2) = \tilde{N}(j-1) = \tilde{N}^*. \ \square\end{aligned}$$

Cut the list from $R(\tilde{N})$ given earlier in (2.1) at the boundary 1's to form rows

$$(1\ \ 1),\ \ (1\ \ 2\ \ 1),\ \ (1\ \ 3\ \ 2\ \ 3\ \ 1),\ \ (1\ \ 4\ \ 3\ \ 5\ \ 2\ \ 5\ \ 3\ \ 4\ \ 1),\ \ldots,$$

where we keep the leftmost 1 for symmetry. Each row, after the first, includes the list of $R(\tilde{N})$ for the preceding row, interspersed with sums of successive pairs of adjacent terms from the preceding row:

$$(1,\ 1),\ (1,\ 1+1,\ 1),\ (1,\ 2+1,\ 2,\ 2+1,\ 1),\ (1,\ 1+3,\ 3,\ 3+2,\ 2,\ 2+3,\ 3,\ 3+1,\ 1),\ \ldots.$$

We recognize the first four lines of Stern's Diatomic array. Our n^{th} row, 1, n, $(n-1)$, $\ldots$, contains 1 followed by the number of Fibonacci representations $R(\tilde{N})$ for consecutive even-Zeck integers $\tilde{N}, F_{2n} \le \tilde{N} \le F_{2n+1}-1$ where $R(F_{2n}) = n, n \ge 1$. We next prove that this array is indeed the same as Stern's diatomic array. Lemma 2.3, which allows us to shift subscripts, was Hoggatt's conjecture and was proved by Klarner [8, Thm. 4]. Lemma 2.4 is part of Lemma 11 from [4].

Lemma 2.3: If sequence $\{b_n\}$ satisfies the Fibonacci recurrence $b_{n+2} = b_{n+1} + b_n$, then $R(b_k - 1) = R(b_{k+1} - 1)$ for k sufficiently large.

Lemma 2.4: Let N be an integer whose Zeckendorf representation has $F_{2c}, c \ge 2$, as its smallest term. Then $R(N) = R(N-1) + R(N+1)$.

Theorem 2.1: Let the n^{th} row of an array list the number of Fibonacci representations $R(\tilde{N})$ for consecutive even-Zeck Integers $\tilde{N}$, $F_{2n} \le \tilde{N} \le F_{2n+1} - 1$. Let $b_{n,k}$ denote the k^{th} term of the n^{th} row, $n = 1, 2, 3, \ldots$, where $b_{n,1} \equiv 1$, and $b_{n,k} = R(\tilde{N}(j_{n,k}))$ for $j_{n,k} = 2^{n-1} + k - 2, k = 2, 3, \ldots, 2^{n-1}+1$. Then $b_{n,k} = a_{n-1,k}$, the k^{th} term in the $(n-1)^{st}$ line in Stern's diatomic array, $n = 1, 2, \ldots$, and $k = 1, 2, \ldots, 2^{n-1}+1$.

Proof: Interpret the leftmost column ($k = 1$) of 1's as $R(F_{2n-1} - 1) = 1$, where $F_{2n-1} - 1$ is the even-Zeck integer preceding F_{2n} according to Lemma 2.2 (iii) with $\tilde{N} = F_{2n}, n \ge 1$. In particular, $b_{1,1} = 1 = a_{0,1}$, and $b_{1,2} = 1 = a_{0,2}$. We show that the two arrays have the same rule of formation by establishing

$$b_{n,2m} = b_{n-1,m} + b_{n-1,m+1} \text{ and } b_{n,2m-1} = b_{n-1,m},\ \ n \ge 2. \tag{2.2}$$

(a) We first prove $b_{n,2m} = b_{n-1,m} + b_{n-1,m+1}$ for $n \ge 2$, $m = 1, \ldots, 2^{n-2}$. The case $m = 1$ is satisfied because $b_{n-1,1} = 1$ by definition, and $b_{n-1,2} = R(\tilde{N}(2^{n-2})) = R(F_{2(n-1)}) = n-1$ from above. For $m = 2, \ldots, 2^{n-2}, \tilde{N}(j) \equiv \tilde{N}(j_{n,2m}) = F_{2n} + \cdots + F_{2c+2p} + F_{2c}, c \ge 2, 1 \le p \le n-3$, when $n \ge 3$ because $c = 1(F_2)$ is not present for even $k \le 2m$ in row n. Also, $n \ge c+p$; hence, $1 \le p \le n-c \le n-2$ for $n \ge 3$; if $n = 2, p = 0$ and $c = 2$. From Lemma 2.2, $\tilde{N}^* = \tilde{N}(j-1)$, $\tilde{N} = \tilde{N}(j) \equiv \tilde{N}(j_{n,2m})$, and $(\tilde{N}+1) = \tilde{N}(j+1)$ are consecutive even-Zeck integers. Hence $b_{n,2m-1} = R(\tilde{N}^*), b_{n,2m} = R(\tilde{N})$, and $b_{n,2m+1} = R(\tilde{N}+1)$ are consecutive entries in the n^{th} row. Since $\tilde{N}(j-1)$ and $\tilde{N}(j+1)$ are each a term in some Fibonacci sequence, apply Lemma 2.3 to shift subscripts down 2 in the expressions for $R(\tilde{N}(j+1))$ and $R(\tilde{N}(j-1))$.

$$\begin{aligned}R(\tilde{N}(j+1)) &= R(\tilde{N}+1) = R\left(F_{2n}+\cdots+F_{2c+2p}+F_{2c}+(F_3-1)\right)\\ &= R((F_{2n-2}+\cdots+F_{2c+2p-2}+F_{2c-2}+F_1)-1)\\ &= R(F_{2(n-1)}+\cdots+F_{2c+2p-2}+F_{2c-2}) = R(\tilde{M}),\end{aligned} \tag{2.3}$$

which is in the $(n-1)^{st}$ row, and $\tilde{M}$, the argument given above, is an even-Zeck integer. The binary representation of $\tilde{M} = F_{2n-2}+\cdots+F_{2c+2p-2}+F_{2c-2}$, $c \geq 2$, is obtained from the binary representation of $\tilde{N}$ by a right-shift by one position (see Lemma 2.1). Because $\tilde{N} \equiv \tilde{N}(j_{n,2m})$, one therefore finds $\tilde{M} \equiv \tilde{N}(j_{n-1,m+1})$; hence by definition $R(\tilde{M}) = b_{n-1,m+1}$. From Lemma 2.2 (iii) (with $c \to c-1 \geq 1$), and Lemma 2.3,

$$\begin{aligned}R(\tilde{N}(j-1)) &= R(\tilde{N}^*) = R(\tilde{N}-F_{2c-2}-1)\\ &= R(F_{2n-2}+\cdots+F_{2c+2p-2}+F_{2c-2}-F_{2c-4}-1) = R(\tilde{M}^*)\end{aligned} \tag{2.4}$$

where $\tilde{M}^*$, defined as the argument of the last R, is the even-Zeck preceding $\tilde{M}$. Hence, $\tilde{M}^* = \tilde{N}(j_{n-1,m})$, and by definition, $R(\tilde{M}^*) = b_{n-1,m}$. What we have to prove now is $R(\tilde{N}) = R(\tilde{M}^*) + R(\tilde{M})$. For this we want to use Lemma 2.4 with $N \to \tilde{N}$. We know already that $R(\tilde{N}+1) = R(\tilde{M})$ but $\tilde{N}-1$ is not an even-Zeck integer for $c \geq 2$. However, we now show that $R(\tilde{N}-1) = R(\tilde{M}^*)$.

$$\begin{aligned}R(\tilde{N}-1) &= R(F_{2n}+\cdots+F_{2c+2p}+F_{2c}-1)\\ &= R(F_{2n-2c+2}+\cdots+F_{2p+2}+F_2-1) = R(\tilde{K})\end{aligned}$$

by shifting subscripts down $(2c-2)$. Recalculate $R(\tilde{N}^*)$ as

$$\begin{aligned}R(\tilde{N}(j-1)) &= R(F_{2n}+\cdots+F_{2c+2p}+F_{2c}-F_{2c-2}-1)\\ &= R((F_{2n}+\cdots+F_{2c+2p}+F_{2c-1}-1)\\ &= R(F_{2n-2c+2}+\cdots+F_{2p+2}+F_1-1) = R(\tilde{K}),\end{aligned}$$

by shifting subscripts down $(2c-2)$. Thus,

$$R(\tilde{N}-1) = R(\tilde{N}(j-1)) = R(\tilde{M}^*). \tag{2.5}$$

Therefore, $b_{n,2m} = b_{n-1,m} + b_{n-1,m+1}$, and part (a) of the proof is finished.

(b) We prove $b_{n,2m-1} = b_{n-1,m}$ for $n \geq 2, m = 1, \ldots, 2^{n-2}$. For $m = 1, b_{n,1} = 1 = b_{n-1,1}$ by definition. If $m = 2, \ldots, 2^{n-2}, b_{n,2m-1} = R(\tilde{N}(j_{n,2m-1})) = R(\tilde{N}^*)$ if we use the same notation as in part (a) of the proof. There we have already shown $R(\tilde{N}^*) = R(\tilde{M}^*) = b_{n-1,m}$, which finishes part (b) of the proof. Together with the input $b_{1,1} = 1 = b_{1,2}$ we have shown that $b_{n,k} = a_{n-1,k}, n = 1, 2, \ldots,$ and $k = 1, 2, \ldots, 2^{n-1} + 1$. □

Corollary 2.1.1: If $\tilde{N}$ is an even-Zeck integer such that Zeck $\tilde{N}$ ends in $F_{2c}, c \geq 2$, and if $\tilde{N}^*$ is the preceding even-Zeck integer, then $R(\tilde{N}^*) = R(\tilde{N} - 1)$. Also, $R(\tilde{N}(j-1)) = R(\tilde{N}(j) - 1)$ with $\tilde{N} = \tilde{N}(j)$.

Proof: See equations (2.4) and (2.5). □

Theorem 2.2: Let $\tilde{N} = \tilde{N}(j)$ be the j^{th} even-Zeck integer, $j = 0, 1, 2, \ldots,$ with $\tilde{N}(0) = 0$. If $R(\tilde{N}) = b_{n,k}$ with $b_{n,k}$ defined in Theorem 2.1, then $\tilde{N} = \tilde{N}(j_{n,k})$ with $j_{n,k} = 2^{n-1} + k - 2, k = 1, 2, \ldots, 2^{n-1} + 1$, for $n = 1, 2, \ldots$. $\tilde{N}(j), j \geq 1$, is obtained by replacing powers 2^p in the dual representation of $j_{n,k}$ by $F_{2(p+1)}$; if $j = 0$, then $\tilde{N} = 0$. Alternately, $\tilde{N} = F_{2n} + \tilde{K}(k-2)$, where $\tilde{K}(k-2)$ is the $(k-2)^{nd}$ even-Zeck integer.

Proof: Apply Lemma 2.1 to Theorem 2.1. □

To illustrate Theorem 2.2, $R(\tilde{N}) = 7 = b_{5,8}$ appears as the 8^{th} term in the 5^{th} row; $n - 2 = 8 - 2 = 6 = 2^2 + 2^1$, yielding $\tilde{N} = F_{2 \cdot 5} + F_{2(2+1)} + F_{2(1+1)} = F_{10} + F_6 + F_4 = 66$, and $R(66) = 7$. The earlier $R(\tilde{N}) = 7 = b_{5,4}$ in that row occurs for $\tilde{N} = F_{10} + F_4 = 58$.

Since the n^{th} row of the array for $R(\tilde{N})$ is the $(n-1)^{st}$ line of Stern's array, several properties of Fibonacci representations of even-Zeck integers $\tilde{N}$ correspond directly to properties given for elements of Stern's diatomic array from Section 1.

1. There are 2^{n-1} even-Zeck integers $\tilde{N}$ in the interval $F_{2n} \leq \tilde{N} \leq F_{2n+1} - 1$. There are $2^{n-1} + 1$ terms $R(\tilde{N})$ in the n^{th} row, whose sum is $3^{n-1} + 1$.
2. The table of $R(\tilde{N})$ values is symmetric; in the n^{th} row, the k^{th} term equals the $(2^{n-1} + 2 - k)^{th}$ term. Compare with $R(F_{2n} + M) = R(F_{2n+1} - 2 - M), 0 \leq M \leq F_{2n-1},\ n \geq 2$, formed from Theorem 1 of [4] by replacing n with $2n$.
3. Dyads $R(\tilde{N})$ correspond to Zeck $\tilde{N}$ ending in $F_{2c}, c \geq 2$; excepting the first column, non-dyads $R(\tilde{N})$ have Zeck $\tilde{N}$ ending in $F_2 = 1$. The dyads have even term numbers.
4. For even-Zeck $\tilde{N}$, consecutive values for $R(\tilde{N})$ are relatively prime. Consecutive values for even-Zeck integers $\tilde{N}$ appear in relatively prime pairs, (3,4), (8,9), (11,12), (21, 22), (24,25), ...

The largest value [2] for $R(\tilde{N})$ in row n is F_{n+1}, corresponding to F_{n+2} as the largest dyad in the n^{th} line as given by Lehmer [9]. Notice that Lemma 2.3 appears in the table as the columns of constants, and the central term in each row is 2. Properties 5 and 6 are explored in the next section.

3. STERN'S DIATOMIC ARRAY APPLIED TO FIBONACCI REPRESENTATIONS

We can find many even-Zeck integers $\tilde{N}$ having a specified value for $R(\tilde{N})$ by applying Theorem 1.1. According to Lehmer [9], Stern gives Euler's $\Phi(m)$ as the number of times that an element m appears in the $(m-1)^{st}$ and all succeeding lines of Stern's diatomic array; this, of course, is our m^{th} row, where values for $R(\tilde{N})$ are the elements, and Euler's $\Phi(m)$ is the number of integers not exceeding m and prime to m. We express $R(\tilde{N})$ as the sum of a pair of relatively prime integers r_1 and r_2, and then use the Euclidean algorithm to write quotients used in the continued fraction for r_1/r_2. The row and column numbers for $R(\tilde{N}) = b_{n,k}$, as well as the Zeckendorf representation of $\tilde{N}$, can be written from those same quotients.

Theorem 3.1: Let $R(\tilde{N}) = b_{n,k}$ as in Theorem 2.1. Let $R(\tilde{N}) = r_1 + r_2$, r_1 and r_2 relatively prime. Use the Euclidean algorithm to write $r_1 = q_1 r_2 + r_3$, $r_2 = q_2 r_3 + r_4, r_3 = q_3 r_4 + r_5, \ldots, r_{m-2} = q_{m-2} + r_{m-1} + r_m$, $r_m = 1$. Then $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, a regular continued fraction. The dyad value $R(\tilde{N})$ occurs in row n, where

$$n = q_1 + q_2 + q_3 + \cdots + q_{m-2} + r_{m-1} + 1; \tag{3.1}$$

$R(\tilde{N})$ occurs between r_1 and r_2, in columns k and $(2^{n-1} + 2 - k)$, $k \geq 2$, where

$$k = 2^{q_1+q_2+\cdots+q_{m-2}+1} - 2^{q_1+q_2+\cdots+q_{m-3}+1} + \cdots - 2^{q_1+q_2+1} + 2^{q_1+1}, m \text{ odd}, \tag{3.2a}$$

$$\text{or } k = 2^{q_1+q_2+\cdots+q_{m-2}+1} - 2^{q_1+q_2+\cdots+q_{m-3}+1} + \cdots - 2^{q_1+1} + 2, m \text{ even}, \tag{3.2b}$$

Proof: Equation (3.1) is (1.2), adjusted by adding 2, since our row numbers are one more than Stern's line numbers, and we are one row farther out. Equation (3.2a) is (1.3) when m is odd, taken one row farther out; k is twice the column number of r_1 in the $(n-1)^{st}$ row. If r_1 is a dyad and thus has an even column number, let $r_2 = b_{n-1,2w+1}$. If r_1 is the left neighbor of r_2, then $r_1 = b_{n-1,2w}$ and $b_{n,k} = r_1 + r_2 = b_{n,2(2w)}$; k is twice the column number of r_1 as (3.2a). If r_1 is the right neighbor of r_2, then $r_1 = b_{n-1,2w+2}$, and $b_{n,k} = r_2 + r_1 = b_{n-1,2w+1} + b_{n-1,2w+2} = b_{2(2w+1)} = b_{n,4w+2}$, so that k is 2 more than twice the column number of r_1 as in (3.2b). □

Lemma 3.1: Let $b_{n,k}$ be the k^{th} term of the n^{th} row of the array of Theorem 2.1. The term directly below $b_{n,k}$ in the $(n+p)^{th}$ row is $b_{n,k} = b_{n+p,2^p(k-1)+1}$. In particular

$$b_{1,1} = b_{1+(n-1),2^{n-1}(1-1)+1} = b_{n,1} = 1, n \geq 1;$$

$$b_{1,2} = b_{1+(n-1),2^{n-1}(2-1)+1} = b_{n,2^{n-1}+1} = 1, n \geq 1;$$

$$b_{p,2} = b_{p+(n-p),2^{n-p}(2-1)+1} = b_{n,2^{n-p}+1} = p, n \geq p, p = 1, 2, \ldots. \tag{3.3}$$

Proof: Lemma 3.1 restates Theorem 1 from [9]: If N has rank R_n in the n^{th} line, it appears directly below in the $(n+k)^{th}$ line with rank $R_{n+k} = 2^k(R_n - 1) + 1$. □

Define a zigzag path through the array of Theorem 2.1 as movement down and right alternating with movement down and left. Define $ZR(y)$ as a movement down y rows and right 1 term; $ZL(x)$, down x rows and left 1 term. From Lemma 3.1,

$$\begin{aligned} ZR(y) &: b_{w,t} \to b_{w+y,[2^y(t-1)+1]+1} = b_{w+y,2^y(t-1)+2}, \\ ZL(x) &: b_{w,t} \to b_{w+x,[2^x(t-1)+1]-1} = b_{w+x,2^x(t-1)}. \end{aligned} \tag{3.4}$$

Lemma 3.2: Let $R(\tilde{N}) = b_{n,k} = r_1 + r_2, r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}], r_{m-1} \geq 2$, where r_1 is a dyad, $r_1 > r_2$. If $b_{n,k}$ is on the left side of the table, the zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(q_{m-2}) \ldots ZR(q_2)ZL(q_1)ZR(1)$, where r_1 is on the left of r_2, m is odd, and k is given by (3.2a); or, $ZR(r_{m-1} - 1)ZL(q_{m-2}) \ldots ZR(q_1)ZL(1)$, where r_1 is on the right of r_2, m is even, and k is given by (3.2b).

If $b_{n,k}$ is on the right side of the table, the zigzag path from $b_{1,2}$ to $b_{n,k}$ is $ZL(r_{m-1} - 1)ZR(q_{m-2}) \ldots ZL(q_2)ZR(q_1)ZL(1)$, for r_1 on the right of r_2, m odd; or, $ZL(r_{m-1} - 1)ZR(q_{m-2}) \ldots ZL(q_1)ZR(1)$, r_1 on the left of r_2, m even. $R(\tilde{N}) = b_{n,2^{n-1}+2-k}$ for k as in (3.2a) or (3.2b) as m is odd or even.

Proof: On the left side of the table, the path from $b_{1,1}$ begins $ZR(r_{m-1} - 1)$ to $b_{r_{m-1},2}$ followed by $ZL(q_{m-2})$. If r_1 is on the left of r_2, the path from $b_{1,1}$ will end with a move $ZR(1)$ to $R(\tilde{N})$, preceded by $ZL(a_1)$ to r_1; m is odd. If r_1 is on the right of r_2, the path from $b_{1,1}$ to $b_{n,k}$ ends $\ldots ZR(q_1)ZL(1)$, so that m is even. Suppose $r_1/r_2 = [a_1; a_2, a_3, r_{m-1}]$. The zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(a_3)ZR(a_2)ZL(a_1)ZR(1)$:

$$\begin{aligned} b_{1,1} \to b_{r_{m-1},2} &\to b_{a_3+r_{m-1},2^{a_3}(2-1)+0} \to b_{a_2+a_3+r_{m-1},2^{a_2}(2^{a_3}-1)+2} \\ &\to b_{a_1+a_2+a_3+r_{m-1},2^{a_1}(2^{a_2+a_3}-2^{a_2}+2-1)+0} \\ &\to b_{a_1+a_2+a_3+r_{m-1}+1,2(2^{a_1+a_2+a_3}-2^{a_1+a_2}+2^{a_1}-1)+2} \\ &= b_{n,2^{a_1+a_2+a_3+1}-2^{a_1+a_2+1}+2^{a_1+1}}; \end{aligned}$$

k is given by (3.2a), $a_1 = q_i$, $i = 1, 2, 3$. This pattern continues for m odd. Suppose $r_1/r_2 = [a_1; a_2, r_{m-1}]$. The zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(a_2)ZR(a_1)ZL(1)$:

$$\begin{aligned} b_{1,1} \to b_{r_{m-1},2} &\to b_{a_2+r_{m-1},2^{a_2}(2-1)+0} \to b_{a_1+a_2+r_{m-1},2^{a_1}(2^{a_2}-1)+2} \\ &\to b_{a_1+a_2+r_{m-1}+1,2(2^{a_1+a_2}-2^{a_1}+2-1)+0} = b_{n,2^{a_1+a_2+1}-2^{a_1+1}+2}; \end{aligned}$$

k is given by (3.2b), $a_i = q_i$, $i = 1, 2$. The pattern continues for m even.

The situation on the right side of the table is similar. The path from $b_{1,2}$ to $b_{n,k}$ on the right side is the mirror image of the path from $b_{1,1}$ to $b_{n,k}$ on the left. □

Lemma 3.3: Let $R(\tilde{N}) = b_{n,k} = r_1 + r_2, r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, $r_{m-1} \geq 2$. If r_1 is the left neighbor or r_2 in the $(n-1)^{st}$ row, and m is odd (even), the ordered sequence, r_1, $R(\tilde{N}), r_2$, appears in the n^{th} row on the left (right) side of the table.

Theorem 3.2 generalizes the zigzag paths of Lemma 3.2 to $\mathfrak{RLRL}\ldots$ patterns, where $\mathfrak{R}(q)$ means to write the next (q) even-subscripted Fibonacci numbers; $\mathfrak{L}(q)$, omit the next (q) even-subscripts. Note that r_1 and r_2 are not ordered.

Theorem 3.2: Let the dyad $R(\tilde{N}) = r_1 + r_2, r_1$ and r_2 relatively prime, appear in the n^{th} row as in Theorem 3.1. If $R(\tilde{N})$ is between r_1 and r_2 on the left side of the table, Zeck $\tilde{N}$ is given from $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}], r_{m-1} \geq 2$, by the $\mathfrak{RLRL}\ldots$ pattern

$$\mathfrak{R}(1)\mathfrak{L}(r_{m-1} - 1)\mathfrak{R}(q_{m-2})\mathfrak{L}(q_{m-3}) \ldots \mathfrak{R}(q_1)\mathfrak{L}(1), m \text{ odd}; \tag{3.5}$$

$\ldots\mathfrak{R}(q_2)\mathfrak{L}(q_1)\mathfrak{L}(1)$, m even. The first Fibonacci number written is F_{2n}.

Proof: Let 2^q correspond to $F_{2(p+1)}$ as in Lemma 2.1; $R(\tilde{N}) = b_{n,k}$ is the term appearing $(k-2)$ entries to the right of $b_{n,2} = R(F_{2n})$ where $F_{2n} = \tilde{N}(2^{n-1})$. From (3.1) with (3.2a) or (3.2b), the highest power of 2 in k has exponent $(q_1 + q_2 + \cdots + q_{m-2}) = (n - r_{m-1} - 1)$. From (3.2a),

$$\begin{aligned}
k-2 &= (2^{q_1+q_2+\cdots+q_{m-2}+1} - 2^{q_1+q_2+\cdots+q_{m-3}+1} + \cdots + (2^{q_1+q_2+q_3+1} - 2^{q_1+q_2+1}) + (2^{q_1+1} - 2) \\
&= 2^{q_1+q_2+\cdots+q_{m-3}+1}(2^{q_{m-2}} - 1) + \cdots + 2^{q_1+q_2+1}(2^{q_3} - 1) + 2(2^{q_1} - 1) \\
&= 2^{q_1+q_2+\cdots+q_{m-3}+1}(2^{q_{m-2}-1} + \cdots + 2 + 1) + \cdots + 2^{q_1+q_2+1}(2^{q_3-1} + \cdots + 2 + 1) \\
&\quad + 2(2^{q_1-1} + \cdots + 1)
\end{aligned}$$

which contains q_{m-2} consecutive powers of 2 beginning with $2^{q_1+q_2+\cdots+q_{m-2}}$, followed by q_{m-3} consecutive missing powers of 2, followed by q_{m-4} consecutive powers of $2, \ldots$, ending with q_1 consecutive powers of 2, with the one final term 2^0 missing. (Recall that k is even, since $R(\tilde{N})$ is a dyad.) In the sum $(2^{n-1} + (k-2))$, the leading exponent in each block of consecutive powers of 2 results from successively subtracting $r_{m-1}, q_{m-2}, q_{m-3}$, from $(n-1)$. If m is even, $(k-2)$ as calculated from (3.2b) ends with $\cdots + (2^{q_1} + 2) - 2$, or $(q_1 + 1)$ missing powers of 2; note that 2^0 is always missing. The pattern of (3.5) follows from Theorems 2.2 and 3.1, and Lemma 3.2. □

Corollary 3.2.1: The zigzag path in which all quotients are 1 leads to $b_{n,k} = R(\tilde{N}) = F_{n+1}$, for $\tilde{N} = F_{2n} + F_{2n-4} + F_{2n-8} + \ldots$, with smallest term F_6 or F_4, as n is odd or even.

Proof: Rewrite $[1; 1, 1, \ldots, 1, 1, 1]$ as $[1; 1, 1, \ldots, 1, 2]$ and use Theorem 3.2. On the right side, $\tilde{N} = F_{2n} + F_{2n-2} + F_{2n-6} + \ldots$, which results from (3.5) if $r_{m-1} = 1$. □

Corollary 3.2.2: If $R(\tilde{N})$ from Theorem 3.2 is between r_1 and r_2 on the right side of the table, Zeck $\tilde{N}$ is written from the $\mathfrak{R}\mathfrak{R}\mathfrak{L}\mathfrak{R}\mathfrak{L}\ldots$ pattern, $\mathfrak{R}(1)\mathfrak{R}(r_{m-1} - 1)\mathfrak{L}(q_{m-2})\mathfrak{R}(q_{m-3})\ldots\mathfrak{R}(q_1)\mathfrak{L}(1)$, m even; or, ending $\ldots\mathfrak{L}(q_1)\mathfrak{L}(1)$, m odd; $r_{m-1} \geq 2$.

Proof: The zigzag path from $b_{1,2}$ to $b_{n,k}$ on the right side is the mirror image of that from $b_{1,1}$ to $b_{n,k}$ on the left side. Recall that $b_{n,k} = b_{n,2^{n-1}+2-k}$ by symmetry. □

To illustrate, compute $\tilde{N}$ from $R(\tilde{N}) = 27 = 19 + 8$. $19/8 = [2; 2, 1, 2]$; $n = (2 + 2 + 1 + 2) + 1 = 8$, $m = 3$. We are on the left side, and Zeck $\tilde{N}$ begins $F_{16}; b_{8,2} = 8$. Interpret the pattern $\mathfrak{R}(1)\mathfrak{L}(2-1)\mathfrak{R}(1)\mathfrak{L}(2)\mathfrak{R}(2)\mathfrak{L}(1)$ as use 16; omit 14; use 12; omit 10 and 8; use 6 and 4; omit 2. Thus, Zeck $\tilde{N} = F_{16} + F_{12} + F_6 + F_4 = 1142$; $R(1142) = 27$. The sequence 19, 27, 8, occurs with $27 = b_{8,40} = R(\tilde{N}(j_{n,k}))$ for $j_{n,k} = 2^{8-1} + (2^5 + 2^2 + 2^1) - 2$, verifying

$\tilde{N} = F_{2(7+1)} + F_{2(5+1)} + F_{2(2+1)} + F_{2(1+1)}$. On the right side, Corollary 3.2.2 gives the associated

solution $\tilde{N}'$ from $\mathfrak{R}(1)\mathfrak{R}(2-1)\mathfrak{L}(1)\mathfrak{R}(2)\mathfrak{L}(2)\mathfrak{L}(1)$ as $\tilde{N}' = F_{16} + F_{14} + F_{10} + F_8 = 1440$, where $R(1440) = 27 = b_{8,90}, 8 = b_{8,89}$ and $19 = b_{8,91}$.

The symmetries of the array for $R(\tilde{N})$ let us find other even-Zeck integers $\tilde{M}$ such that $R(\tilde{M}) = R(\tilde{N})$, with $R(\tilde{M})$ and $R(\tilde{N})$ both appearing in the n^{th} row. Theorem 3.3 gives a special solution for $\tilde{M}$.

Theorem 3.3: Let $R(\tilde{N}) = r_1 + r_2, r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, as in Theorem 3.2; $q_1 \geq 1, r_{m-1} \geq 2$. Let Zeck $\tilde{M}$ be written from the $\mathfrak{R}\mathfrak{L}\mathfrak{R}\mathfrak{L}\ldots$ pattern of (3.5), adjusted by taking the quotients of r_1/r_2 in ascending order: $\mathfrak{R}(1)\mathfrak{L}(q_1)\mathfrak{R}(q_2)\mathfrak{L}(q_3)\ldots\mathfrak{R}(r_{m-1}-1)\mathfrak{L}(1)$, m odd; $\ldots\mathfrak{R}(q_{m-2})\mathfrak{L}(r_{m-1}-1)\mathfrak{L}(1)$, m even. Then $R(\tilde{M}) = R(\tilde{N})$, both appearing in row n.

Proof: A reversal identity for continued fractions appears as Theorem 1 in [1]: if $[a_0, a_1, \ldots, a_{n-1}, a_n] = p_n/q_n$, then $[a_n, a_{n-1}, \ldots, a_1, a_0\} = p_n/p_{n-1}$. Here, $R(\tilde{M}) = p_n = R(\tilde{N})$. □

Theorem 3.3 applied to the preceding example gives $\mathfrak{R}(1)\mathfrak{L}(2)\mathfrak{R}(2)\mathfrak{L}(1)\mathfrak{R}(2-1)\mathfrak{L}(1)$ or $\tilde{M} = F_{16} + F_{10} + F_8 + F_4 = 1066$; $R(1066) = 27$, but $\tilde{N} = 1142$.

The Calkin and Wilf [6] sequence (1.4) is the same as our sequence (2.1); that is, $b(j) = R(\tilde{N}(j))$, where $b(j)/b(j+1)$ is the j^{th} rational number, $j = 0, 1, 2, \ldots$. Thus, the results of the present paper allow us to write the j^{th} rational number. Given j, by Lemma 2.1, we can write Zeck $\tilde{N}(j)$, the Zeckendorf representation of the j^{th} even-Zeck integer; there are several ways [4] to compute $R(\tilde{N}(j))$ and $R(\tilde{N}(j+1))$. Given any rational number a/b, Theorem 3.2 can be adapted to find $\tilde{N}(j)$ such that $a/b = R(\tilde{N}(j))/R(\tilde{N}(j+1))$. For example, to answer at which position the rational number 13/8 appears, place 13 between 5 and 8 in the n^{th} row, $5, 13, 8; r_1/r_2 = 5/8 = [0; 1, 1, 1, 2], n = 6$, m is even. Since $R(\tilde{N}(j)) = 13$ is on the right side of the table, Corollary 3.2.2 gives $\tilde{N}(j) = F_{12} + F_{10} + F_6 = 207$, and $\tilde{N}(j+1) = 208$, where $R(207) = 13$, $R(208) = 8$. From Zeck $\tilde{N}(j)$, $j = 2^{12/2-1} + 2^{10/2-1} + 2^{6/2-1} = 2^5 + 2^4 + 2^2 = 52$; thus, 13/8 is the 52^{nd} rational number. Another example: to find 5/12, use the sequence $5, 12, 7; 5/7 = [0; 1, 2, 2], n = 6, m = 3$. We are on the left side; Theorem 3.2 gives $\tilde{N}(j+1) = F_{12} + F_8 + F_6 = 173; R(173) = 12$. The preceding even-Zeck integer $\tilde{N}(j) = F_{12} + F_8 + F_4 + F_2 = 169, R(169) = 5; j = 2^{12/2-1} + 2^{8/2-1} + 2^{4/2-1} + 2^{2/2-1} = 43$. Thus, 5/12 is the 43^{rd} rational number. We note that $R(\tilde{N}(j))$ is another function $f(j)$ such that $f(j)/f(j+1)$ takes every rational value exactly once, answering a question posed in [6].

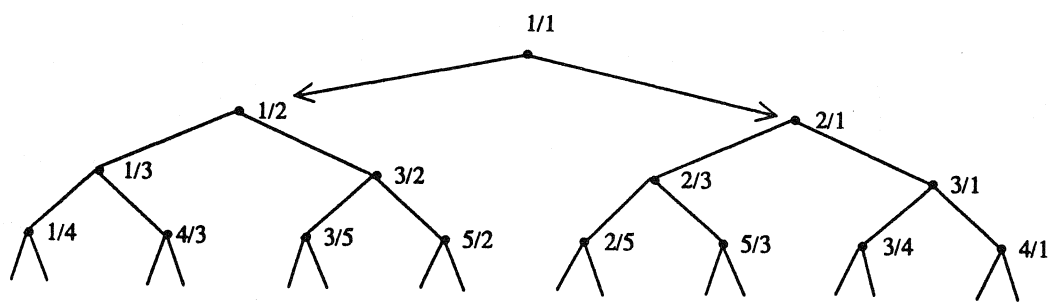

Figure 1. The Calkin-Wilf version of the tree of fractions

Further, we can write the address of the rational number r_1/r_2 appearing in Calkin and Wilf's tree of fractions, which is a variant of the Stern-Brocot tree [7]. The tree of fractions (Figure 1) has 1/1 at the top of the tree. Each vertex r_1/r_2 has two children; its left child is $r_1/(r_1+r_2)$, and its right child is $(r_1+r_2)/r_2$; each fraction is $R(\tilde{N}(j))/R(\tilde{N}(j+1))$ for some j. In the n^{th} row of the tree, the numerators are the first 2^{n-1} terms of our n^{th} row. Let $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}], r_m = 1, q_1 \geq 0, r_{m-1} \geq 2$; if m is odd (even), r_1/r_2 appears on the left (right) side of the tree, and r_1 is on the left (right) of r_2 in the table. Starting from 1/1, if m is odd, the vertex r_1/r_2 has the address $L^{r_{m-1}-1}R^{q_{m-2}}\ldots L^{q_2}R^{q_1}$; if m is even, $R^{r_{m-1}-1}L^{q_{m-2}}\ldots R^{q_2}L^{q_1}$; where L^q means to move q vertices left; R^q, move q vertices right; L^0 and R^0 are not written. If r_1 is the left neighbor of r_2 in the table and $R(\tilde{N}(j)) = r_1+r_2$, then $R(\tilde{N}(j))/R(\tilde{N}(j+1))$ is the right child of r_1/r_2; if instead $R(\tilde{N}(j+1)) = r_1+r_2$, then $R(\tilde{N}(j))/R(\tilde{N}(j+1))$ is the left child of r_1/r_2.

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REFERENCES

[1] A.T. Benjamin, F.E. Su, and J. Quinn. "Counting on Continued Fractions." *Mathematics Magazine* **73.2** (2000): 98-104.

[2] M. Bicknell-Johnson. "The Smallest Positive Integer Having F_k Representations as Sums of Distinct Fibonacci Numbers." *Applications of Fibonacci Numbers*, Vol. **8** Dordrecht: Kluwer 1999: pp. 47-52.

[3] M. Bicknell-Johnson. "The Least Integer Having p Fibonacci Representations, p Prime." *The Fibonacci Quarterly* **40.3** (2002): 260-265.

[4] M. Bicknell-Johnson, and D.C. Fielder. "The Number of Representations of N Using Distinct Fibonacci Numbers, Counted by Recursive Formulas." *The Fibonacci Quarterly* **37.1** (1999): 47-60.

[5] M. Bicknell-Johnson, and D.C. Fielder. "The Least Number Having 331 Representations as a Sum of Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **39.5** (2001): 455-461.

[6] N. Calkin, and H.S. Wilf. "Recounting the Rationals." *The American Mathematical Monthly* **107** (2000): 360-363.

[7] R.L. Graham, E. Knuth, and O. Patashnik. *Concrete Mathematics* Reading: Addison-Wesley, 1999. Chapters 4, 5, 6.

[8] D.A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **6.4** (1968): 235-44.

[9] D.H. Lehmer. "On Stern's Diatomic Series." *American Mathematical Monthly* **36.2** (1929): 59-67.

[10] N.J.A. Sloane. "On-Line Encyclopedia of Integer Seuqence." http://www.research.att.com /ñjas/sequences/.

[11] M.A. Stern. "Über eine zahlentheoretische Funkton." *J. Reine Angew. Math.* **55** (1858): 193-220.

[12] S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Ellis Horwood, Ltd., 1989. Chap. IX.

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✠ ✠ ✠

Tribute

JoAnn Vine

JoAnn Vine, *Fibonacci Quarterly* typist for 25 years, is retiring. She never missed a deadline and hates to give it up, but it is time to retire.

JoAnn sang with the San Francisco Opera before she married Richard Vine (*FQ* Subscription Manager for 17 years). She started her statistical typing business in 1964, typing theses for students at Stanford and San Jose State.

Thank you, JoAnn, for your years of dedicated service to the Fibonacci Association!