ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-599 Proposed by the Editor

For every $n \geq 0$ let $C_n := \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. Show that all the solutions of the diophantine equation $F_m = C_n$ have $m \leq 5$.

H-600 Proposed by Arulappah Eswarathasan, Hofstra University, Hempstead, NY

The Pseudo-Fibonacci numbers u_n are defined by $u_1 = 1$, $u_2 = 4$ and $u_{n+2} = u_{n+1} + u_n$. A number of the form $3s^2$, where s is an integer, is called a one-third square. Show that $u_0 = 3$ and $u_{-4} = 12$ are the only one-third squares in the sequence.

H-601 Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

Prove or disprove that the sequence

$$\left\{\frac{\sqrt[n]{L_2 \cdot \ldots \cdot L_{n+1}}}{\alpha^{(n+3)/2}}\right\}_{n \geq 1}$$

strictly decreases to its limit 1. Here, α is the golden section.

H-602 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

Find the limit

$$\lim_{n \to \infty} \frac{\Gamma\left(-k\frac{F_{n+1}}{\alpha F_n}\right)}{\Gamma\left(-\ell\frac{L_{n+1}}{\alpha L_n}\right)},$$

where k and ℓ are fixed positive integers, Γ is the Euler function, and α is the golden section.

SOLUTIONS

Sums of consecutive Fibonacci numbers

H-588 Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Barcelona, Spain

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} F_{k+2} \geq \frac{n^{n+1}}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{L_{k+1}^{\frac{n+1}{n}} - F_{k+1}^{\frac{n+1}{n}}}{L_{k+1} - F_{k+1}} \right\}.$$

Solution by H.-J. Seiffert, Berlin, Germany

Direct computation shows that equality holds with $n = 1$. Now, suppose that $n \geq 2$. If a and b are real numbers such that $b > a > 0$, then, by Hölder's Inequality,

$$\int_a^b t^{\frac{1}{n}} dt \leq \left(\int_a^b t dt \right)^{\frac{1}{n}} \left(\int_a^b dt \right)^{\frac{n-1}{n}},$$

or, equivalently,

$$\frac{n}{n+1} \cdot \frac{b^{\frac{n+1}{n}} - a^{\frac{n+1}{n}}}{b-a} \leq \left(\frac{a+b}{2} \right)^{\frac{1}{n}}.$$

Applying this inequality with $a := F_{k+1}$ and $b := L_{k+1}$, noting that $F_{k+1} + L_{k+1} = 2F_{k+2}$, and taking the product over $k = 1, \ldots, n$, gives

$$\frac{n^n}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{L_{k+1}^{\frac{n+1}{n}} - F_{k+1}^{\frac{n+1}{n}}}{L_{k+1} - F_{k+1}} \right\} \leq \left(\prod_{k=1}^{n} F_{k+2} \right)^{\frac{1}{n}}.$$

By the Arithmetic-Geometric Mean Inequality, we have

$$n \left(\prod_{k=1}^{n} F_{k+2} \right)^{\frac{1}{n}} \leq \sum_{k=1}^{n} F_{k+2},$$

and the desired inequality follows.

Also solved by Paul Bruckman, Walther Janous and the proposers.

Iterated Fibonacci numbers

H-589 Proposed by Robert DiSario, Bryant College, Smithfield, RI

Let $f(n) = F(F(n))$, where $F(n)$ is the n^{th} Fibonacci number. Show that

$$f(n) = \frac{(f(n-1))^2 - (-1)^{F(n)}(f(n-2))^2}{f(n-3)}$$

for $n > 3$.

Solution by L.A.G. Dresel, Reading, England

We shall first prove the identity $F_{s+t}F_{s-t} = (F_s)^2 - (-1)^{s-t}(F_t)^2$, which corresponds to formula I (19) on page 59 of [1]. Using $\alpha\beta = -1$ and the Binet form for F_n, we have

$$5F_{s+t}F_{s-t} = (\alpha^{s+t} - \beta^{s+t})(\alpha^{s-t} - \beta^{s-t}) = \alpha^{2s} + \beta^{2s} - (\alpha\beta)^{s-t}(\alpha^{2t} + \beta^{2t})$$

$$= \alpha^{2s} - 2(\alpha\beta)^s + \beta^{2s} - (\alpha\beta)^{s-t}\{\alpha^{2t} - 2(\alpha\beta)^t + \beta^{2t}\} = 5\{(F_s)^2 - (-1)^{s-t}(F_t)^2\}.$$

Putting $s := F_{n-1}$ and $t := F_{n-2}$, we have $s + t = F_n$ and $s - t = F_{n-3}$, so that our identity takes the form $f(n)f(n-3) = (f(n-1))^2 - (-1)^{F(n-3)}(f(n-2))^2$. But since $F_n = 2F_{n-2} + F_{n-3}$ we have $(-1)^{F(n-3)} = (-1)^{F(n-1)}$, and for $n > 3$ we can divide by $f(n-3)$, which proves the given formula.

1. V.E. Hoggatt. "Fibonacci and Lucas numbers." Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by P. Bruckman, M. Catalani, O. Furdui, W. Janous, H. Kwong, V. Mathe, H.-J. Seiffert, J. Spilker and the proposer.

Arithmetic Functions of Fibonacci Numbers

H-590 Proposed by Florian Luca, IMATE, UNAM, Morelia, Mexico

For any positive integer k let $\phi(k)$, $\sigma(k)$, $\tau(k)$, $\Omega(k)$, $\omega(k)$ be the Euler function of k, the sum of divisors function of k, the number of divisors function of k, and the number of prime divisors function of k (where the primes are counted with or without multiplicity), respectively.

1. Show that $n|\phi(F_n)$ holds for infinitely many n.
2. Show that $n|\sigma(F_n)$ holds for infinitely many n.
3. Show that $n|\tau(F_n)$ holds for infinitely many n.
4. Show that for no $n > 1$ can n divide either $\omega(F_n)$ or $\Omega(F_n)$.

Solution by J.-Ch. Schlage-Puchta & J. Spilker, Albert-Ludwigs-Universität Freiburg, Germany

We first prove a

Lemma: Let $f : \boldsymbol{N} \to \boldsymbol{Z}$ be multiplicative such that $f(p^k)$ is even for all primes $p > 2$ and all odd positive integers k. Then $2^n|f(F_{2^n})$ holds for every $n \geq 6$.

Examples:

1. The Euler function ϕ is multiplicative and $\phi(p^k) = p^{k-1}(p-1)$ is even if $p > 2$. This is part 1 of the Problem.

2. The sum of jth powers of divisors function $\sigma_j(n) = \sum_{d|n} d^j$, $j \geq 0$ is multiplicative and $\sigma_j(p^k) = 1 + p^j + \cdots + p^{(k-1)j}$ is even if both p and k are odd. The cases $j = 1$ and $j = 0$ are parts 2 and 3 of the Problem, respectively.

Proof of the Lemma: Define the multiplicative function

$$f^*(p^k) := \begin{cases} 1, & \text{if } p = 2, \\ f(p^k), & \text{if } p > 2. \end{cases}$$

Then $f^*(2^k n) = f^*(n)$ and $f^*(n)|f(n)$ hold for all positive integers k and n. It suffices to show that

(1) $64|f^*(F_{64})$;

(2) if $n > 6$ and $n|f^*(F_n)$, then $2n|f^*(F_{2n})$.

Claim (1) above follows from the fact that F_{64} is odd, squarefree, and has precisely 6 prime factors. For Claim (2) above, we use the facts that $F_{2n} = F_n L_n$ and $L_n^2 - 5F_n^2 = (-1)^n \cdot 4$. From the last formula, it follows that $\gcd(F_n, L_n)|2$. Thus, writing $2^a||F_n$ and $2^b||L_n$, we get

$$f^*(F_{2n}) = f^*(F_n L_n) = f^*\left(\frac{F_n}{2^a} \cdot \frac{L_n}{2^b}\right) = f^*\left(\frac{F_n}{2^a}\right) f^*\left(\frac{L_n}{2^b}\right) = f^*(F_n) f^*(L_n).$$

By the hypotheses of the Lemma, $f^*(L_n)$ is always even except when L_n is a square or twice times a square. A result from [1] says that the only such values of n are $n = 1,\ 3,\ 6$. Thus, if $n > 6$, then $f^*(L_n)$ is even, which completes the proof (2) and of the Lemma.

For part 4 of the Problem, assume that $n > 2$. Then $F_n > 1$, and so on the one hand writing the prime factorization of F_n we get

$$F_n = \prod_j p_j^{k_j} \geq \prod_j 2^{k_j} = 2^{\sum_j k_j},$$

while on the other hand, by the Binet formula, we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^n \left(\frac{1 - (\beta/\alpha)^n}{\sqrt{5}}\right) < \alpha^n \cdot \frac{2}{\sqrt{5}} < \alpha^n < 2^n,$$

where α is the golden section and β is its conjugate. Thus, $n > \sum_j k_j = \Omega(F_n) \geq \omega(F_n)$, which shows that n cannot divide neither $\omega(F_n)$ nor $\omega(F_n)$.

1. J.H.E. Cohn. "Lucas and Fibonacci numbers and some Diophantine equations." Proc. Glasgow Math. Assoc. 7 (1965): 24-28.

Editor's Remark: All solutions used powers of 2 with exponent greater than or equal to 6 to settle parts 1–3 of the problem, and quoted the result from [1] above to the effect that L_n is a perfect square only for $n = 1,\ 3$. However, one does not need the full strength of the result from [1] in this instance. Indeed, since $L_2 = 3$ and $L_{2^n} = L_{2^{n-1}}^2 - 2$ holds for all $n \geq 2$, it follows easily, by induction, that $L_{2^n} \equiv 3 \pmod 4$ holds for all $n \geq 1$, and as such these numbers cannot be perfect squares.

Also solved P. Bruckman, V. Mathe and the proposer.

Please Send in Proposals!