HEIGHTS OF HAPPY NUMBERS AND CUBIC HAPPY NUMBERS

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1. INTRODUCTION

Let $S_2 : \mathbf{Z}^+ \to \mathbf{Z}^+$ denote the function that takes a positive integer to the sum of the squares of its decimal digits. For $a \in \mathbf{Z}^+$, let $S_2^0(a) = a$ and for $m \geq 1$ let $S_2^m(a) = S_2(S_2^{m-1}(a))$. A *happy number* is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$. It is well known that 4 is not a happy number and that, in fact, for all $a \in \mathbf{Z}^+$, a is not a happy number if and only if $S_2^m(a) = 4$ for some $m \geq 0$. (See [4] for a proof.) The *height* of a happy number is the least $m \geq 0$ such that $S_2^m(a) = 1$. Hence, 1 is a happy number of height 0; 10 is a happy number of height 1; and 7 is a happy number of height 5.

Similarly, we define $S_3 : \mathbf{Z}^+ \to \mathbf{Z}^+$ to be the function that takes a positive integer to the sum of the cubes of its decimal digits. A *cubic happy number* is a positive integer a such that $S_3^m(a) = 1$ for some $m \geq 0$. The height of a cubic happy number is defined in the obvious way. So, 1 is a cubic happy number of height 0; 10 is a cubic happy number of height 1; and 112 is a cubic happy number of height 2.

By computing the heights of each happy number less than 400, it is straightforward to find the least happy numbers of heights up to 6. (These, as well as the least happy number of height 7, can also be found in [2] and [5].) Richard Guy [3] reports that Jud McCranie verified the value of the least happy number of height 7 and determined the value of the least happy number of height 8. Guy further reports that Warut Roonguthai determined the least happy number of height 9. These results and their methods of proof have not, to the best of our knowledge, appeared in the literature.

The goal of this paper is to present a method for confirming these and additional results. Along with determining the least happy number of height 10 and providing proofs for other happy numbers of small heights, we find with proof the least cubic happy numbers of small heights. Our algorithms combine computer and by-hand calculations. It should be noted that none of the computer calculations took special packages beyond the usual C++ language and no program needed longer than a few seconds to run.

In Section 2 we present the least happy numbers of heights 0 through 10 and describe the methods used to determine them. In Section 3 we do the same for the least cubic happy numbers of heights 0 through 8.

2. SQUARING HEIGHTS

Table 1 gives the least happy numbers of heights 0 through 10. Those through height 7 are easily found by simply iterating S_2 on each positive integer up to about 80,000, until 1 or 4 is reached and recording the number of iterations needed when 1 is attained. The goal of this section is to explain both the derivations and the proofs for the rest of the table.

height	happy number
0	1
1	10
2	13
3	23
4	19
5	7
6	356
7	78999
8	$3789 \times 10^{973} - 1$
9	$78889 \times 10^{(3789 \times 10^{973} - 306)/81} - 1$
10	$259 \times 10^{[78889 \times 10^{(3789 \times 10^{973} - 306)/81} - 13]/81} - 1$

Table 1: The least happy numbers of heights 0-10.

As described above, we used a simple computer search to determine the heights of all happy numbers less than 80,000. The only other computer routine we used in this work is a nested search in which we checked when a fixed number was equal to the sum of squares of a certain number of single digit integers. (See below for more details.) Since the number of single digit integers is never large, the search takes very little time.

To prove that $3789 \times 10^{973} - 1$ is the least happy number of height 8, we begin with a lemma that is immediate from the first computer search mentioned above.

Lemma 1: *The only happy numbers of height 7 less than* 80,000 *are* 78999, 79899, 79989, *and* 79998.

Theorem 2: *The least happy number of height* 8 *is*

$$\sigma_8 = 3789 \times 10^{973} - 1 = 3788\overbrace{99\ldots9}^{973}.$$

Proof. It is easy enough to check that $3789 \times 10^{973} - 1$ is indeed a happy number of height 8. To prove that it is least, let $x \le 3789 \times 10^{973} - 1$ be a happy number of height 8. Then $S_2(x) < 3^2 + 976 \times 9^2 = 79065$. $S_2(x)$ must be a happy number of height 7 so, by Lemma 1, $S_2(x) = 78999$. We see that x must have at least 973 9's in its base 10 expansion since otherwise $S_2(x) < 3^2 + 4 \times 8^2 + 972 \times 9^2 < 78999$. Assuming without loss of generality that the digits of x are in nondecreasing order, we have that

$$x = a_1a_2a_3a_4\overbrace{99\ldots9}^{973},$$

with $0 \le a_1 \le 3$, and $a_1 \le a_2 \le a_3 \le a_4 \le 9$. Since $S_2(x) = 78999$, we have $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 78999 - 973 \times 9^2 = 186$. A computer search through the possible values of a_1, a_2, a_3, and a_4 finds that the only solution is

$$x = 3788\overbrace{99\ldots9}^{973} = \sigma_8,$$

as desired. □

The proofs for the least happy numbers of heights 9 and 10 follow the same general outline. We get as a corollary to the above proof that the only happy numbers of height 8 less than 4×10^{976} are numbers whose digits are permutations of the digits of σ_8. This result will take the place of Lemma 1 in the proof of Theorem 3.

Theorem 3: *The least happy number of height* 9 *is*

$$\sigma_9 = 78889 \times 10^{(3789 \times 10^{973} - 306)/81} - 1 = 78888 \overbrace{999\ldots99}^{(\sigma_8 - 305)/81}.$$

Proof. Again, we can easily verify that this is a happy number of the desired height. Let $x \leq \sigma_9$ be a happy number of height 9, with the digits of x in nondecreasing order. Then $S_2(x) < 7^2 + [(\sigma_8 - 305)/81 + 4] \times 9^2 < \sigma_8 + 68$ and so, from the previous proof, $S_2(x) = \sigma_8$. Now, x must have at least $(\sigma_8 - 305)/81$ 9's in its base 10 expansion since otherwise $S_2(x) < 7^2 + 5 \times 8^2 + [(\sigma_8 - 305)/81 - 1] \times 9^2 < \sigma_8$. So we have

$$x = a_1 a_2 a_3 a_4 a_5 \overbrace{999\ldots99}^{(\sigma_8 - 305)/81},$$

with $0 \leq a_1 \leq 7$, and $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq 9$. Since $S_2(x) = \sigma_8$, we have $\sum_{i=1}^{5} a_i^2 = 305$. A computer search shows that the only solution is $x = \sigma_9$. □

Theorem 4: *The least happy number of height* 10 *is*

$$\sigma_{10} = 259 \times 10^{(78889 \times 10^{3789 \times 10^{973} - 306)/81} - 94]/81} - 1 = 258 \overbrace{999\ldots99}^{(\sigma_9 - 93)/81}.$$

Proof. σ_{10} is a happy number of height 10, since $S_2(\sigma_{10}) = \sigma_9$. Let $x \leq \sigma_{10}$ be a happy number of height 10, with nondecreasing digits. Then $S_2(x) < 2^2 + [(\sigma_9 - 93)/81 + 2] \times 9^2 \leq \sigma_9 + 73$ and so, from the previous proof, $S_2(x) = \sigma_9$. We see that x must have at least $(\sigma_9 - 93)/81 - 4$ 9's in its base 10 expansion since otherwise $S_2(x) < 2^2 + 7 \times 8^2 + [(\sigma_9 - 93)/81 - 5] \times 9^2 < \sigma_9$. So we have

$$x = a_1 a_2 a_3 a_4 a_5 a_6 a_7 \overbrace{9999\ldots99}^{(\sigma_9 - 93)/81 - 4},$$

with $0 \leq a_1 \leq 2$, and $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq 9$. Since $S_2(x) = \sigma_9$, we have $\sum_{i=1}^{7} a_i^2 = 417$. A computer search shows that $x = \sigma_{10}$. □

This method can certainly be extended to find additional least happy numbers of given heights. There are a few obstacles to be dealt with. The main one is simply finding a good candidate for the least happy number, thus bounding the size of the numbers being considered. It's always possible to find a happy number of a given height: simply take a happy number of the next smaller height and string that many 1's in a row. More efficient is taking as many 9's as possible, then adding digits as needed to obtained the desired sum. This explains the divisions by 81 that appear in the expressions for σ_9 and σ_{10}.

The problem is in finding a *good* candidate for the least happy number of a given height. If the candidate is too large, then the size of the search through sums of squares may be prohibitive. Further, it may be impossible to prove that the image under S_2 of the least happy number of the desired height must be the least happy number of the next smaller height or even a permutation of its digits. This could lead to the need for separate searches for additional happy numbers of smaller heights.

It may be that these two problems are not solvable by finding a good candidate. Theoretically, it may be that even with the least happy number found, the search through sums of squares may take too long. It would be interesting to know if there is a bound on the size of the search, that is on the number of unknown digits, regardless of the height involved.

3. CUBING HEIGHTS

In this section, we apply the methods developed in Section 2 to the problem of finding least cubic happy numbers of given heights. Table 2 gives the least cubic happy numbers of heights 0 through 8.

height	cubic happy number
0	1
1	10
2	112
3	1189
4	778
5	13477
6	$238889 \times 10^{16} - 1$
7	$1127 \times 10^{3276941015089163237} - 1$
8	$35678 \times 10^{(1127 \times 10^{3276941015089163237} - 1055)/729} - 1$

Table 2: The least cubic happy numbers of height 0-8.

As in the last section, we start by computing the heights of all cubic happy numbers less than 20,000. This gives us the least cubic happy numbers of heights 0 through 5. To save computing time we use the fact that all cubic happy numbers are congruent to 1 modulo 3, since S_3 preserves congruence classes modulo 3. In the case of happy numbers, we used the fact that a positive integer x is not a happy number if and only if, for some $m \geq 0$, $S_2^m(x) = 4$. For the cubic case, we use the fact that a positive integer congruent to 1 modulo 3 is not a cubic happy number if and only if, for some $m \geq 0$, $S_3^m(x) \in \{55, 136, 160, 370, 919\}$. (See [1] or [2].)

Our method of proof is basically as in Section 2. We consider a positive integer x less than or equal to our claimed least cubic happy number and prove that x must, in fact, equal our candidate. Obviously, in place of a search through sums of squares, we do a computer search through sums of cubes. Otherwise, the algorithm is the same.

Again, we begin with a lemma that is immediate from the computation finding the heights of cubic happy numbers less than 20,000.

Lemma 5: *The only cubic happy numbers of height 5 less than 16,000 are 13477, 13747, 13774, 14377, 14737, and 14773.*

Theorem 6: *The least cubic happy number of height* 6 *is*

$$\gamma_6 = 238889 \times 10^{16} - 1 = 238888\overbrace{9\ldots9}^{16}.$$

Proof. Since $S_3(\gamma_6) = 13747$ is a cubic happy number of height 5, γ_6 is a cubic happy number of height 6. Let $x \le \gamma_6$ be a cubic happy number of height 6 with digits in nondecreasing order. Then $S_3(x) < 2^3+21\times 9^3 = 15317$. Since $S_3(x)$ must be a cubic happy number of height 6, by Lemma 5, we have that $S_3(x) \in \{13477, 13747, 13774, 14377, 14737, 14773\}$. Further, x must have at least 13 9's in its base 10 expansion since otherwise $S_3(x) < 2^3+9\times 8^3+12\times 9^3 = 13364$ which is too small. This gives us that

$$x = a_1a_2a_3a_4a_5a_6a_7a_8a_9\,\overbrace{99\ldots 9}^{13},$$

with $0 \le a_1 \le 2$, and $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \le a_9 \le 9$. We search for combinations such that $\sum_{i=1}^{9} a_i^3 + 13 \times 9^3 \in \{13477, 13747, 13774, 14377, 14737, 14773\}$. A computer search shows that the only solution is

$$x = 238888\,\overbrace{9\ldots 9}^{16} = \gamma_6. \ \square$$

Theorem 7: *The least cubic happy number of height 7 is*

$$\gamma_7 = 1127 \times 10^{3276941015089163237} - 1 = 1126\,\overbrace{9999\ldots 99}^{(\gamma_6-226)/729}.$$

Proof. It's easy to verify that γ_7 is a cubic happy number of height 7. Let $x \le \gamma_7$ be a cubic happy number of height 7 with digits in nondecreasing order. Then $S_3(x) < 1^3 + [(\gamma_6 - 226)/729 + 3] \times 9^3 < \gamma_6 + 1962$. From the computer search in the previous proof, it follows that $S_3(x) = \gamma_6$. Now, x must have at least $(\gamma_6 - 226)/729 - 6$ 9's in its base 10 expansion since otherwise $S_3(x) < 1^3 + 10 \times 8^3 + [(\gamma_6 - 226)/729 - 7] \times 9^3 < \gamma_6$. So

$$x = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}\,\overbrace{99999\ldots 999}^{(\gamma_6-226)/729-6},$$

with $0 \le a_1 \le 1$, and $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \le a_9 \le a_{10} \le 9$. Since $S_3(x) = \gamma_6$, we need $\sum_{i=1}^{10} a_i^3 = 4600$. A computer search shows that the only solution is $x = \gamma_7$. $\square$

Theorem 8: *The least cubic happy number of height 8 is*

$$\gamma_8 = 35678 \times 10^{(1127\times 10^{3276941015089163237}-1055)/729} - 1 = 35677\,\overbrace{9999\ldots 999}^{(\gamma_7-1054)/729}.$$

Proof. As usual, we start by noting that γ_8 is indeed a cubic happy number of height 8. Now, let $x \le \gamma_8$ be a cubic happy number of height 8 with digits in nondecreasing order. Then $S_3(x) < 3^3 + [(\gamma_7 - 1054)/729 + 4] \times 9^3 = \gamma_7 + 1889$. From the computer search in the previous proof, it follows that $S_3(x) = \gamma_7$. Now, x must have at least $(\gamma_7 - 1054)/729 - 4$ 9's in its base 10 expansion since otherwise $S_3(x) < 3^3 + 9 \times 8^3 + [(\gamma_7 - 1054)/729 - 5] \times 9^3 < \gamma_7$. So

$$x = a_1a_2a_3a_4a_5a_6a_7a_8a_9\,\overbrace{99999\ldots 999}^{(\gamma_7-1054)/729-4},$$

with $0 \le a_1 \le 3$, and $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \le a_9 \le 9$. Since $S_3(x) = \gamma_7$, we need $\sum_{i=1}^{9} a_i^3 = 3970$. A computer search shows that the only solution is $x = \gamma_8$. $\square$

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