UNEXPECTED PELL AND QUASI MORGAN-VOYCE SUMMATION CONNECTIONS

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1. PRELIMINARIES

Motivation

In [1], the connection between *Pell convolution numbers* $P_n^{(m)}$ and *Quasi Morgan-Voyce polynomials* $S_n^{(r,u)}(x)$ was established.

Here, the objective is to display a set of nine neat formulas (Theorems 1-9) expressing $S_n^{(r,u)}(x)$ in terms of finite sums involving $P_n^{(m)}(r,u=0,1,2)$ with $P_0^{(m)}=0$, while $P_n^{(m)}(n<0)$ is not defined. Central to this theme is the germinal polynomial $S_n^{(1,1)}(x)$.

Initially, the impetus for this paper originated from a nice result (Theorem 1) discovered by J.M. Mahon [3], [4], to whom indebtedness is gratefully acknowledged.

Background Material

Firstly, note that [1, (3.3)]

$$S_n^{(r,u)}(x)=\sum_{k=0}^{n} d_{n,k}^{(r,u)}x^k \tag{1.1}$$

with certain restrictions [1, (3.4)] on $d_{n,k}^{(r,u)}$. Secondly [1, Theorem 1],

$$d_{n,0}^{(r,u)}=P_nr+\frac{1}{2}Q_nu \tag{1.2}$$

where P_n, Q_n are the Pell and Pell-Lucas numbers [2], respectively, with $P_n\equiv P_n^{(0)}, Q_n\equiv Q_n^{(0)}$.
Allusion to (1.1) and (1.2) will be constantly made.

Results (1.3) - (1.6) are required in the demonstration of proofs:

$$P_n^{(m)}=2P_{n-1}^{(m)}+P_{n-2}^{(m)}+P_n^{(m-1)} \qquad (recurrence) \tag{1.3}$$

$$P_{n+1-m}^{(m)}+P_{n-1-m}^{(m)}=\frac{n}{m}P_{n+1-m}^{(m-1)} \tag{1.4}$$

$$P_n^{(m)}-P_{n-1}^{(m)}=\frac{n+2m-1}{2m}P_n^{(m-1)} \tag{1.5}$$

$$d_{n,k}^{(r,u)}=P_{n+1-k}^{(k-1)}+P_{n-k}^{(k)}r+\frac{n-k}{2k}P_{n+1-k}^{(k-1)}u. \tag{1.6}$$

Results (1.3) - (1.6) occur in [1] as [1, (2.1)], [1, (2.4)], [1, (2.5)], and [1 Theorem 4], respectively. They are necessary tools of trade in this paper.

Please notice the correction in (1.6) to the first factor in the third term in the enunciation of [1, Theorem 4], namely, $\frac{n-k}{2k}$ instead of $\frac{n-2}{2^k}$

Guiding Comments

(i) Familarity with [1, Table 1] and [1, Table 2] is essential.
(ii) Choice of $n = 3$ in all the Examples of the Theorems provides some basis for comparison.
(iii) Because of the variety of approaches available in the proofs, some detail of all proofs is appropriate.
(iv) Generally (Theorems 3-9), the technique for developing the proofs lies in "spotting" the involvement of two or more $S_n^{(1,1)}(x)$ and hence pursuing the corresponding arithmetic for the $d_{n,k}^{(r,u)}$.

2. THE SUMMATION RESULTS

Theorem 1 (Mahon [3]): $S_n^{(1,1)}(x) = \sum_{k=0}^{n} P_{n+1-k}^{(k)} x^k$.

Proof: Now $S_n^{(1,1)}(x) = \sum_{k=0}^{n} d_{n,k}^{(1,1)} x^k$ by (1.1).
But

$$
\begin{aligned}
d_{n,k}^{(1,1)} &= P_{n+1-k}^{(k-1)} + P_{n-k}^{(k)} + \frac{n-k}{2k} P_{n+1-k}^{(k-1)} \text{ by (1.6)} \\
&= P_{n+1-k}^{(k-1)} + P_{n+1-k}^{(k)} - \frac{(n-k+1+2k-1)}{2k} P_{n+1-k}^{(k-1)} + \frac{n-k}{2k} P_{n+1-k}^{(k-1)} \text{ by (1.5)} \\
&= P_{n+1-k}^{(k)},
\end{aligned}
$$

whence the theorem follows by (1.1).

Alternative Proof [4]: Use induction on n in conjunction with (1.3).

Example: $S_3^{(1,1)}(x) = 12 + 14x + 6x^2 + x^3 = P_4^{(0)} + P_3^{(1)} x + P_2^{(2)} x^2 + P_1^{(3)} x^3$.

Theorem 2: $S_n^{(0,0)}(x) = \sum_{k=0}^{n-1} P_{n-k}^{(k)} x^{k+1}$.

Proof:

$$
\begin{aligned}
S_n^{(0,0)}(x) &= x\boldsymbol{B}_n(x) \quad \text{by [1, (4.5)]} \\
&= xS_{n-1}^{(1,1)}(x) \quad \text{by [1, (4.1)]},
\end{aligned}
$$

where $\boldsymbol{B}_n(x)$ is the quasi Morgan-Voyce analogue [1] of the corresponding standard Morgan-Voyce polynomial $B_n(x)$. Theorem 2 is thus an immediate consequence of Theorem 1.

Example: $S_3^{(0,0)}(x) = 5x + 4x^2 + x^3 = P_3^{(0)} x + P_2^{(1)} x^2 + P_1^{(2)} x^3$.

Theorem 3: $S_n^{(0,1)}(x) = \sum_{k=1}^n \frac{n+k}{2k} P_{n+1-k}^{(k-1)} x^k + \frac{Q_n}{2}$.

Proof: Consider $S_n^{(1,1)}(x) - S_{n-1}^{(1,1)}(x)$. Then

$$d_{n,k}^{(1,1)} - d_{n-1,k}^{(1,1)} = P_{n+1-k}^{(k-1)} + P_{n-k}^{(k)} + \frac{n-k}{2k} P_{n+1-k}^{(k-1)}$$

$$- P_{n-k}^{(k-1)} - P_{n-1-k}^{(k)} - \frac{n-1-k}{2k} P_{n-k}^{(k-1)} \quad \text{by (1.6)}$$

$$= P_{n+1-k}^{(k-1)} + \frac{n-k+2k-1}{2k} P_{n-k}^{(k-1)} + \frac{n-k}{2k} P_{n+1-k}^{(k-1)}$$

$$- P_{n-k}^{(k-1)} - \frac{n-1-k}{2k} P_{n-k}^{(k-1)} \quad \text{by (1.5)}$$

$$= P_{n+1-k}^{(k-1)} + \frac{n-k}{2k} P_{n+1-k}^{(k-1)} = \frac{n+k}{2k} P_{n+1-k}^{(k-1)} \qquad (\alpha)$$

$$= d_{n,k}^{(0,1)} \quad \text{by (1.6).}$$

Invoking (α) ensures the theorem. Be aware that the isolated Pell-Lucas term $\frac{1}{2}Q_n$ arises when $k = 0$. Also see (1.2) for $r = 0,\ u = 1$.

Example: $S_3^{(0,1)}(x) = 7 + 10x + 5x^2 + x^3 = \frac{1}{2}Q_3 + 2P_3^{(0)}x + \frac{5}{4}P_2^{(1)}x^2 + P_1^{(2)}x^3$.

Theorem 4: $S_n^{(1,0)}(x) = \sum_{k=1}^n \left(P_{n-k}^{(k)} + P_{n+1-k}^{(k-1)}\right) x^k + P_n$.

Proof: Consider $S_{n-1}^{(1,1)}(x)$ for k, added to $S_{n-1}^{(1,1)}(x)$ for $k-1$. So

$$d_{n-1,k}^{(1,1)} + d_{n-1,k-1}^{(1,1)} = P_{n-k}^{(k-1)} + P_{n-1-k}^{(k)} + \frac{n-1-k}{2k} P_{n-k}^{(k-1)}$$

$$+ P_{n+1-k}^{(k-2)} + P_{n-k}^{(k-1)} + \frac{n-k}{2(k-1)} P_{n+1-k}^{(k-2)} \quad \text{by (1.6)}$$

$$= P_{n-1-k}^{(k)} + P_{n-k}^{(k-1)} + \frac{n-1+k}{2k} P_{n-k}^{(k-1)} + \frac{n-k}{2(k-1)} P_{n+1-k}^{(k-2)} \quad \text{by (1.5)}$$

$$= P_{n-k}^{(k)} + P_{n+1-k}^{(k-1)} \qquad (\beta)$$

$$= d_{n,k}^{(1,0)} \quad \text{by (1.6),}$$

whence the theorem ensues on appeal to (β). Observe that the extraneous Pell number P_n occurs when $k = 0$, in conformity with (1.2) for $r = 1, u = 0$.

Example: $S_3^{(1,0)}(x) = 5 + 9x + 5x^2 + x^3 = P_3 + \left(P_3^{(0)} + P_2^{(1)}\right) x + \left(P_2^{(1)} + P_1^{(2)}\right) x^2 + \left(P_1^{(2)} + P_0^{(3)}\right) x^3$.

Theorem 5: $S_n^{(2,2)}(x) = \sum_{k=1}^{n} \left(2P_{n+1-k}^{(k)} - P_{n+1-k}^{(k-1)}\right) x^k + 2P_{n+1}$.

Proof: Consider $2S_n^{(1,1)}(x)$ for k, then subtract $S_{n-1}^{(1,1)}(x)$ for $k-1$. Accordingly,

$$\begin{aligned}
2d_{n,k}^{(1,1)} - d_{n-1,k-1}^{(1,1)} &= 2P_{n-k}^{(k)} + \frac{n+k}{k} P_{n+1-k}^{(k-1)} - P_{n-k}^{(k-1)} - \frac{n-2+k}{2(k-1)} P_{n+1-k}^{(k-2)} \quad \text{by (1.6)} \\
&= 2P_{n+1-k}^{(k)} - P_{n+1-k}^{(k-1)} \quad \text{by (1.4), (1.5), (1.3) and simplifying} \qquad (\gamma) \\
&= \left(P_{n+1-k}^{(k)} - P_{n-1-k}^{(k)} - P_{n+1-k}^{(k-1)}\right) + P_{n+1-k}^{(k)} + P_{n-1-k}^{(k)} \\
&= 2P_{n-k}^{(k)} + P_{n+1-k}^{(k)} + P_{n-1-k}^{(k)} \quad \text{by (1.3)} \\
&= 2P_{n-k}^{(k)} + \frac{n}{k} P_{n+1-k}^{(k-1)} \quad \text{by (1.4)} \\
&= P_{n+1-k}^{(k-1)} + 2P_{n-k}^{(k)} + 2 \cdot \frac{n-k}{2k} P_{n+1-k}^{(k-1)} \\
&= d_{n,k}^{(2,2)} \quad \text{by (1.6)}.
\end{aligned}$$

Applying (γ), we have the theorem where $2P_{n+1}$ originates with $k = 0$. Refer again to (1.2), where $r = u = 2$ in this case.

Example:

$$\begin{aligned}
S_3^{(2,2)}(x) &= 24 + 23x + 8x^2 + x^3 \\
&= 2P_4 + (2P_3^{(1)} - P_3^{(0)})x + (2P_2^{(2)} - P_2^{(1)})x^2 + (2P_1^{(3)} - P_1^{(2)})x^3.
\end{aligned}$$

Theorem 6: $S_n^{(1,2)}(x) = \sum_{k=0}^{n} \left(P_{n+1-k}^{(k)} + P_{n-k}^{(k)} + P_{n-1-k}^{(k)}\right) x^k$.

Proof: Consider $S_{n,k}^{(1,1)}(x) + S_{n-1,k}^{(1,1)}(x) + S_{n-2,k}^{(1,1)}(x)$, leading to

$$\begin{aligned}
d_{n,k}^{(1,1)} + d_{n-1,k}^{(1,1)} + d_{n-2,k}^{(1,1)} &= P_{n+1-k}^{(k-1)} + \frac{n-k}{2k}P_{n+1-k}^{(k-1)} + P_{n-k}^{(k)} \\
&\quad + P_{n-k}^{(k-1)} + \frac{n-1-k}{2k}P_{n-k}^{(k-1)} + P_{n-1-k}^{(k)} \\
&\quad + P_{n-1-k}^{(k-1)} + \frac{n-2-k}{2k}P_{n-1-k}^{(k-1)} + P_{n-2-k}^{(k)} \\
&= P_{n+1-k}^{(k)} + P_{n-k}^{(k)} + P_{n-1-k}^{(k)} \quad \text{using (1.5) three times} \qquad (\delta) \\
&= P_{n-k}^{(k)} + \frac{n}{k}P_{n+1-k}^{(k-1)} \quad \text{by (1.4)} \\
&= P_{n+1-k}^{(k-1)} + P_{n-k}^{(k)} + \frac{n-k}{2k}P_{(n+1-k)}^{(k-1)} \cdot 2 \\
&= d_{n,k}^{(1,2)} \quad \text{by (1.6),}
\end{aligned}$$

whence the theorem is assured by (δ).

Example: $S_3^{(1,2)}(x) = 19 + 19x + 7x^2 + x^3 = \left(P_4^{(0)} + P_3^{(0)} + P_2^{(0)}\right) + \left(P_3^{(1)} + P_2^{(1)} + P_1^{(1)}\right)x +$ $\left(P_2^{(2)} + P_1^{(2)} + P_0^{(2)}\right)x^2 + \left(P_1^{(3)} + P_0^{(3)} + P_{-1}^{(3)}\right)x^3$.

Outlines of Proofs of Theorems 7-9:

Anticipating that the reader's appetite may have been whetted a little, we hopefully leave the remaining proofs as minor challenges, while giving a indication in each case of the appropriate procedure.

Theorem 7: $S_n^{(0,2)}(x) = \sum_{k=1}^{n} \frac{n}{k} P_{n+1-k}^{(k-1)} x^k + Q_n$.

Proof: This resembles Theorem 3. Use $S_n^{(1,1)}(x) + S_{n-2}^{(1,1)}(x)$ giving

$$\begin{aligned}
d_{n,k}^{(1,1)} + d_{n-2,k}^{(1,1)} &= P_{n+1-k}^{(k)} + P_{n-1-k}^{(k)} \quad \text{by Theorem 1, applied twice} \\
&= \frac{n}{k}P_{n+1-k}^{(k-1)} \quad \text{by (1.4)} \\
&= P_{n+1-k}^{(k-1)} + \frac{n-k}{2k}P_{n+1-k}^{(k-1)} \cdot 2 \\
&= d_{n,k}^{(0,2)} \quad \text{by (1.6).}
\end{aligned}$$

Once again, we recall that the appendage constant term Q_n in the enunciation of the theorem refers to $k=0$, noting that this is guaranteed by (1.2) for $r=0, u=2$.

Example: $S_3^{(0,2)}(x) = 14 + 15x + 6x^2 + x^3 = Q_3 + 3P_3^{(0)}x + \frac{3}{2}P_2^{(1)}x^2 + P_1^{(2)}x^3$.

Theorem 8: $S_n^{(2,0)}(x) = \sum_{k=1}^n \left(2P_{n-k}^{(k)} + P_{n+1-k}^{(k-1)}\right)x^k + 2P_n$.

Proof: This resembles Theorem 4. Use (1.5) and (1.6) to produce

$$\begin{aligned} 2d_{n-1,k}^{(1,1)} + d_{n-1,k-1}^{(1,1)} &= P_{n+1-k}^{(k-1)} + 2P_{n-k}^{(k)} \\ &= d_{n,k}^{(2,0)} \end{aligned}$$

with $k=0$ yielding the exterior term $2P_n$, confirmed by (1.2) for $r=2, u=0$.

Example:

$$\begin{aligned} S_3^{(2,0)}(x) &= 10 + 13x + 6x^2 + x^3 \\ &= 2P_3 + (2P_2^{(1)} + P_3^{(0)})x + (2P_1^{(2)} + P_2^{(1)})x^2 + (2P_0^{(3)} + P_1^{(2)})x^3. \end{aligned}$$

Theorem 9: $S_n^{(2,1)}(x) = \sum_{k=1}^n \left(2P_{n-k}^{(k)} + \frac{n+k}{2k}P_{n+1-k}^{(k-1)}\right)x^k + 2P_n + \frac{1}{2}Q_n$.

Proof: If we consider the simple addition $S_n^{(1,1)}(x) + S_{n-1}^{(1,1)}(x)$, then

$$\begin{aligned} d_{n,k}^{(1,1)} + d_{n-1,k}^{(1,1)} &= 2P_{n-k}^{(k)} + \frac{n+k}{2k}P_{n+1-k}^{(k-1)} \\ &= d_{n,k}^{(2,1)} \end{aligned}$$

eventually, after applying (1.6) and (1.5) and tidying up. Our theorem is then validated, remembering that $d_{n,0}^{(2,1)} = 2P_n + \frac{1}{2}Q_n$ by (1.2).

Example:

$$\begin{aligned} S_3^{(2,1)}(x) &= 17 + 18x + 7x^2 + x^3 \\ &= (10+7) + (2P_2^{(1)} + 2P_3^{(0)})x + (2P_1^{(2)} + \frac{5}{4}P_2^{(1)})x^2 + (2P_0^{(3)} + P_1^{(2)})x^3. \end{aligned}$$

3. AFTERTHOUGHTS

Relationships among (i) the $d_{n,k}^{(r,u)}$, (ii) the $S_n^{(r,u)}(x)$

Simple links connecting the $d_{n,k}^{(r,u)}$, each with a corresponding nexus involving the $S_n^{(r,u)}(x)$, are relatively easy to discover from the material in Theorems 1-9. For convenience we will drop the functional notation for $S_n^{(r,u)}(x)$ in this segment. Thus:

Temporary Convention: $S_n^{(r,u)}(x) \equiv S_n^{(r,u)}$.

Theorems	Connections	
3, 9	$\left.\begin{array}{l} d_{n,k}^{(0,1)} + d_{n,k}^{(2,1)} = 2d_{n,k}^{(1,1)} \\ S_n^{(0,1)} + S_n^{(2,1)} = 2S_n^{(1,1)} \end{array}\right\}$	(3.1)
4, 8	$\left.\begin{array}{l} d_{n,k}^{(2,0)} - d_{n,k}^{(1,0)} = d_{n-1,k}^{(1,1)} \\ S_n^{(2,0)} - S_n^{(1,0)} = S_{n-1}^{(1,1)} \end{array}\right\}$	(3.2)
6, 7	$\left.\begin{array}{l} d_{n,k}^{(1,2)} - d_{n,k}^{(0,2)} = d_{n-1,k}^{(1,1)} \\ S_n^{(1,2)} - S_n^{(0,2)} = S_{n-1}^{(1,1)}. \end{array}\right\}$	(3.3)

Appropriate right-hand sides of (3.2), (3.3) are the same, whereas those of (3.1) are twice as great.

Furthermore, Theorems 5 and 8 together yield

$$\left.\begin{array}{l} d_{n,k}^{(2,2)} + d_{n,k}^{(2,0)} = 2\left(d_{n,k}^{(1,1)} + d_{n-1,k}^{(1,1)}\right) \\ S_n^{(2,2)} + S_n^{(2,0)} = 2\left(S_n^{(1,1)} + S_{n-1}^{(1,1)}\right). \end{array}\right\} \tag{3.4}$$

Verifications of (3.1) - (3.4) may readily be checked for $n = 3$ by using data already provided in the text, along with $S_2^{(1,1)} = 5 + 4x + x^2$.

Lastly, observe that from (1.2),

$$d_{n,0}^{(r,u)} - d_{n,0}^{(u,r)} = (u - r)P_{n-1} = 0 \text{ if } r = u. \tag{3.5}$$

More generally, a quick investigation of $d_{n,k}^{(r,u)} - d_{n,k}^{(u,r)}$ could be undertaken.

FINALE

Our self-contained set of propositions (Theorems 1-9) has been a pleasurable challenge to the author who at no time found himself wandering in "the bloomless meadows of algebra", as envisaged by the character in the novel by Robert Louis Stevenson and Lloyd Osbourne, *The Wrecker*. Moreover, it has exploited the opportunity to expand our knowledge of the coefficients $d_{n,k}^{(r,u)}$ from [1].

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