ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2004. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-966 Proposed by Stanley Rabinowitz, Math Pro, Westford, MA

Find a recurrence relation for $r_n = \frac{1}{1+F_n}$.

B-967 Proposed by Juan Pla, Paris, France

Prove that $\frac{5}{32}F_{6n}^2$ is an integer of the form $\frac{m(m+1)}{2}$.

B-968 Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN

Let $F(n) = \sum_{i=2}^{n} \frac{4+1000F_i}{F_{i-1}F_{i+1}}$ where F_i is the i^{th} Fibonacci number. Find $\lim\limits_{n\to\infty} F(n)$.

B-969 Proposed by José Luis Díaz-Barrero, UPC, Barcelona, Spain

Evaluate the following sum

$$\sum_{n=1}^{\infty} \frac{F_{n+1}[F_{2n+3}+(-1)^{n+1}]F_{n+3}}{F_{n+2}[F_{2n+1}+(-1)^n][F_{2n+5}+(-1)^{n+2}]}.$$

B-970 Proposed by Peter G. Anderson, Rochester Institute of Technology, Rochester, NY

Define a second-order and three third-order recursions by:

$$f_n = f_{n-1} + f_{n-2}, \text{ with } f_0 = 1, f_1 = 1.$$
$$g_n = g_{n-1} + g_{n-3}, \text{ with } g_0 = 1, g_1 = 1, g_2 = 1.$$
$$h_n = h_{n-2} + h_{n-3}, \text{ with } h_0 = 1, h_1 = 0, h_2 = 1.$$

and

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \text{ with } t_0 = 1, t_1 = 1, t_2 = 2.$$

Prove:

1. $t_{n+3} = f_{n+3} + \sum_{p+q=n} f_p t_q$.
2. $t_{n+2} = g_{n+2} + \sum_{p+q=n} g_p t_q$.
3. $t_{n+1} = h_{n+1} + \sum_{p+q=n} h_p t_q$.

SOLUTIONS

Another Fibonacci Sequence

B-951 Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA (Vol. 41, no. 1, February 2003)

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

Solution by Carl Libis, University of Rhode Island, Kingston, RI.

We will show by induction that $u_n = F_{2n-1}/L_{2n-1}$. Note that $u_1 = 1 = F_1/L_1$. Assume that $u_n = F_{2n-1}/L_{2n-1}$. Then

$$u_{n+1} = \frac{3u_n+1}{5u_n+3} = \frac{3\frac{F_{2n-1}}{L_{2n-1}}+1}{5\frac{F_{2n-1}}{L_{2n-1}}+3} = \frac{3F_{2n-1}+L_{2n-1}}{5F_{2n-1}+3L_{2n-1}} = \frac{3F_{2n-1}+(2F_{2n-2}+F_{2n-1})}{5F_{2n-1}+3(2F_{2n-2}+F_{2n-1})}$$

$$= \frac{2(2F_{2n-1}+F_{2n-2})}{2(4F_{2n-1}+3F_{2n-2})} = \frac{F_{2n-1}+F_{2n}}{F_{2n-1}+3F_{2n}} = \frac{F_{2n+1}}{F_{2n+1}+2F_{2n}} = \frac{F_{2n+1}}{F_{2n+2}+F_{2n}} = \frac{F_{2n+1}}{L_{2n+1}}$$

This completes the induction.

All the received solutions used a similar argument. A slight generalization was given by H.J. Seifert.
Also solved by Paul S. Bruckman, Mario Catalani, Charles Cook, Kenneth Davenport, Steve Edwards, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Emrah Kiliç, Harris Kwong, Kathleen Lewis, Reiner Martin, H.-J. Seiffert, James Sellers, J. Spilker, David Stone, J. Suck, Haixing Zhao, and the proposer.

And ... a Fibonacci Identity

B-952 Proposed by Scott H. Brown, Auburn University, Montgomery, AL (Vol. 41, no. 1, February 2003)

Show that

$$10F_{10n-5}+12F_{10n-10}+F_{10n-15}=25F_{2n}^5+25F_{2n}^3+5F_{2n}$$

for all integers $n\geq 2$.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY.

Using the following identities from [1]

$$\begin{array}{ll} (I_{16}) & 5F_{2m}^2=L_{4m}-2, \\ (I_{15}) & L_{2m}^2=L_{4m}+2, \\ (I_{24}) & L_mF_p=F_{m+p}-F_{m-p}, \quad p \text{ even}, \end{array}$$

we find

$$\begin{aligned} 25F_{2n}^5+25F_{2n}^3+5F_{2n} &= F_{2n}[(5F_{2n}^2)^2+5\cdot 5F_{2n}^2+5] \\ &= F_{2n}[(L_{4n}-2)^2+5(L_{4n}-2)+5] \\ &= F_{2n}[L_{4n}^2+L_{4n}-1] \\ &= F_{2n}[L_{8n}+L_{4n}+1] \\ &= (F_{10n}-F_{6n})+(F_{6n}-F_{2n})+F_{2n} \\ &= F_{10n}. \end{aligned}$$

Letting $G_n=F_{5n}$, it suffices to prove that

$$G_{2n}=10G_{2n-1}+12G_{2n-2}+G_{2n-3}, \quad n\geq 2. \tag{1}$$

From the generating function

$$\sum_{n=0}^{\infty}G_nx^n=\frac{1}{\alpha-\beta}\left\{\sum_{n=0}^{\infty}(\alpha^5x)^n-\sum_{n=0}^{\infty}(\beta^5x)^n\right\}=\frac{1}{\alpha-\beta}\left\{\frac{1}{1-\alpha^5x}-\frac{1}{1-\beta^5x}\right\}$$

$$=\frac{1}{\alpha-\beta}\cdot\frac{(\alpha^5-\beta^5)x}{1-(\alpha^5+\beta^5)x+(\alpha^5\beta^5)x^2}=\frac{F_5x}{1-L_5x+(-1)^5x^2}=\frac{5x}{1-11x-x^2},$$

we deduce that $q^2 - q - 1 = 0$ is the characteristic equation for G_n. Hence G_n satisifies the recurrence relation

$$G_n = 11G_{n-1} + G_{n-2}, \quad n \geq 2,$$

from which (1) follows immediately.

Reference

1. Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, pages 56-59, Fibonacci Association, 1969.

1. Walther Janous made the remark that " ... an immeidate consequence of the above identity is that for all $n \geq 2$, $5F_{2n}$ divides $10F_{10n-5} + 12F_{10n-10} + F_{10n-15}$. This suggests the following problem: Determine all 7-tuples (a, b, c, d, A, B, C) where a, b, c, A, B, C are positive integers and d is a non-zero integer such that for all $n \geq \max(0, \frac{-c}{d})$, aF_{bn} divides $AF_{cn+d} + BF_{cn+2d} + CF_{cn+3d}$ and g.c.d$(a, A, B, C) = 1$ and a can not be increased".
2. H.-J. Seiffert prove the identity

$$10F_{k-5} + 12F_{k-10} + F_{k-15} = F_k \text{ for all } K \in Z.$$

Also solved by Paul Bruckman, Mario Catalani, Kenny Davenport, L.A.G. Dresel, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, N. Gauthier, Walther Janous, Emrah Kilis, William Moser, H.-J. Seiffert, J. Suck, Haixing Zhao, and the proposer.

Never Perfect!

B-953 Proposed by Harvey J. Hindin, Huntington Station, NY (Vol. 41, no. 1, February 2003)

Show that

$$(F_n)^4 + (F_{n+1})^4 + (F_{n+2})^4$$

is never a perfect square. Similarly, show that

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

is never a perfect square, when W_n is defined for all integers n by $W_n = pW_{n-1} - qW_{n-2}$ and where $W_0 = a$ and $W_1 = b$.

Solution by H.-J. Seiffert, Berlin, Germany

If $p = q = \sqrt[4]{2}$ and $a = b = 1$, then $W_0 = W_1 = 1$ and $W_2 = 0$, so that $(qW_0)^4 + (pW_1)^4 + (W_2)^4 = 4$ is a perfect square.

Now, suppose that $p, q, a,$ and b are all integers with $pq \neq 0$. Let n be an integer such that the integer $q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2$ is nonzero (this is satisfied for all integers n if $\langle W_n \rangle = \langle F_n \rangle$). Since [1]

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4 = (q^2W_n^2 + p^2W_{n+1}^2 + W_{n+2}^2)^2/2$$

and since $\sqrt{2}$ is irrational, the expression on the left hand side of the above identity then cannot be a perfect square.

Reference

1. R.S. Melham & H. Kwong. "Problem B-927." *The Fibonacci Quarterly* **40.4** (2002): 374-75.

Paul Brukman discussed the Case $pq = 0$ and showed that it lead to trivial solutions. Even when $pq = 0, A = (gW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$ may still be zero for some n, if a and b are properly chosen. To avoid much difficulties, he suggested the addition of " ... is never a $\underline{non - zero}$ perfect square ... " in the statement of the problem.

Also solved by Paul Bruckman, Mario Catalani, L.A.G. Dresel, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Harris Kwong, Carl Libis (1st part), Reiner Martin, and the proposer.

A Fibonacci floor-and-ceiling Equality

B-954 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 41, no. 1, February 2003)

Let n be a nonnegative integer. Show that

$$\sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)} = L_{2\lfloor n/2 \rfloor+1} + \sqrt{5}F_{2\lceil n/2 \rceil},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor-and ceiling-function, respectively.

Solution by L.A.G. Dresel, Reading, England

Let S_n denote the expression on the left side of the proposed identity. Since $\alpha + \beta = 1, \alpha\beta = -1$ and $\alpha^2 = \alpha + 1$, we have $\sqrt{5} + 2 = 2\alpha + 1 = \alpha^2 + \alpha = \alpha(\alpha + 1) = \alpha^3$, so that $S_n = \sqrt{\{\alpha^3(\alpha^{2n+1} - 2 - \beta^{2n+1})\}} = \alpha\sqrt{\{\alpha^{2n+2} - 2\alpha + \beta^{2n}\}} = \alpha\{\alpha^{n+1} - (-1)^n\beta^n\}$, giving $S_n = (1+\alpha)\alpha^n - (-1)^n(1-\beta)\beta^n = \alpha^n + \alpha^{n+1} - (-1)^n(\beta^n - \beta^{n+1})$. Therefore, when n is <u>even</u>, we have $S_n = L_{n+1} + \sqrt{5}F_n$, and when n is <u>odd</u>, we have $S_n = L_n + \sqrt{5}F_{n+1}$, which agrees with the given formula.

Also solved by Paul Bruckman, Mario Catalani, Kenny Davenport, Sergio Falcón and Angel Plaza (jointly), Ovidiu Furdui, Walther Janous, Harris Kwong, Haixing Zhao, and the proposer.

A Strict Inequality

B-955 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

(Vol. 41, no. 1, February 2003)

Prove that

$$1 < \frac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \frac{1}{\sqrt{1+F_{2n+1}^2}} + \frac{1}{\sqrt{1+F_{2n+2}^2}} < \frac{3}{2}$$

for all integers $n \geq 0$.

Solution by Paul S. Bruckman, Berkeley, CA

Let $x_1 = F_{2n}\{1 + (F_{2n})^2\}^{-1/2}, x_2 = \{1 + (F_{2n+1})^2\}^{-1/2}, x_3 = \{1 + (F_{2n+2})^2\}^{-1/2}$, for a given $n \geq 0$. Clearly, $x_1 \geq 0, x_2 \geq x_3 > 0$. Moreover, if $n > 0, x_1 > x_2 > x_3 > 0$. Let $S(n) = x_1 + x_2 + x_3$. Note $S(0) = 0 + 2 \cdot 2^{-1/2} = 2^{1/2} \approx 1.41$, hence $1 < S(0) < 1.5$. Also, $S(1) = 2^{-1/2} + 5^{-1/2} + 10^{-1/2} \approx 1.47$, hence $1 < S(1) < 1.5$.

Next, $S(2) = 3 \cdot 10^{-1/2} + 26^{-1/2} + 65^{-1/2} \approx 1.27$. hence $1 < S(2) < 1.5$.

Henceforth, we suppose $n \geq 3$. Then $S(n) < (F_{2n} + 2)\{1 + (F_{2n})^2\}^{-1/2} < 1 + 2/F_{2n} \leq 1 + 2/F_6 = 1.25$. Hence $S(n) < 1.5$ for all $n \geq 0$. In fact, $S(n) \leq S(1)$ for all $n \geq 0$.

On the other hand, if $n \geq 3$,

$$\begin{aligned}
S(n) &= \{1 + (F_{2n})^{-2}\}^{-1/2} + (F_{2n+1})^{-1}\{1 + (F_{2n+1})^{-2}\}^{-1/2} + (F_{2n+2})^{-1}\{1 + (F_{2n+2})^{-2}\}^{-1/2} \\
&> 1 - 1/\{2(F_{2n})^2\} + (F_{2n+1})^{-1}(1 - 1/\{2(F_{2n+1})^2\}) + (F_{2n+2})^{-1}(1 - 1/\{2(F_{2n+2})^2\}) \\
&= 1 + 1/F_{2n+1} + 1/F_{2n+2} - 1/2\{1/(F_{2n})^2 + 1/(F_{2n+1})^3 + 1/(F_{2n+2})^3\} \\
&> 1 + 2/F_{2n+2} - (F_{2n} + 2)/\{2(F_{2n})^3\} > 1 + 2/F_{2n+2} - 1/(F_{2n})^2.
\end{aligned}$$

Note that $2(F_{2n})^2 - F_{2n+2} > 0$ if $n \geq 3$, hence $2/F_{2n+2} - 1/(F_{2n})^2 > 0$. Therefore, $S(n) > 1$ for all $n \geq 3$. From our previous results, $S(n) > 1$ for all $n \geq 0$.
Q.E.D.

We may also note that $\lim_{n \to \infty} S(n) = 1$.

Also solved by Walther Janous, Angel Plaza and Sergio Falcón (jonitly), and the proposer.