CHAINS OF EQUIVALENT FIBONACCI-WISE TRIANGLES

DEWEY C. DUNCAN, Los Angeles, California

Consider the infinite set of ordered and equally dispersed Fibonacci numbers, F_{n+ih}, $i = 0, 1, 2, \cdots, n, h$, arbitrary positive integers. The triangle having vertices at the points designated by the rectangular cartesian coordinates

(F_n, F_{n+h}), (F_{n+2h}, F_{n+3h}), (F_{n+4h}, F_{n+5h}) has the area

$$F_{2h}F_{5h} + F_hF_{4h} - F_{3h}F_{4h} - F_hF_{2h} ,$$

which is noted to be independent of n and depends only upon the dispersion of the Fibonacci numbers used for coordinates of the vertices.

PROOF

Twice the area of the specified triangle is equal to the absolute value of the determinant

$$\begin{vmatrix} F_n & F_{n+h} & 1 \\ F_{n+2h} & F_{n+3h} & 1 \\ F_{n+4h} & F_{n+5h} & 1 \end{vmatrix}$$

whose expanded form, simplified by the identity

$$F_{a+b} = F_bF_{a+1} + F_{b-1}F_a ,$$

reduces to

$$AF_{n+1}^2 + BF_{n+1}F_n + CF_n^2 ,$$

wherein

$$\begin{aligned} A &= F_{2h}F_{5h} + F_hF_{4h} - F_{3h}F_{4h} - F_hF_{2h} , \\ B &= F_{2h}F_{5h-1} + F_{5h}F_{2h-1} + F_{h-1}F_{4h} + F_{3h} \\ &\qquad - F_{3h}F_{4h-1} - F_{4h}F_{3h-1} - F_hF_{2h-1} - F_{5h} , \\ C &= F_{2h-1}F_{5h-1} + F_{h-1}F_{4h-1} + F_{3h-1} \\ &\qquad - F_{3h-1}F_{4h-1} - F_{h-1}F_{2h-1} - F_{5h-1} . \end{aligned}$$

By use of the identity cited above, the fundamental relationship $F_n + F_{n+1} = F_{n+2}$, one may easily prove that $A = -B = -C$. Furthermore, since

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = \pm 1 ,$$

the area of the triangle is observed to be half the value of

$$F_{2h}F_{5h} + F_hF_{4h} - F_hF_{2h} - F_{3h}F_{4h} . \qquad \text{Q. E. D.}$$

COROLLARIES

1. For any positive integral value of h there are 2h chains of Fibonacci-wise triangles; i.e., triangles of equal area extending along the two series of vertex points whose rectangular cartesian coordinates are equally dispersed Fibonacci numbers. In each chain consecutive triangles have two vertices in common.

2. By exhibiting the fundamental relationship of Fibonacci numbers as $F_{n+1} - F_n = F_{n-1}$, one may define the Fibonacci numbers for zero and negative indices, to wit, $F_0 = 0$, $F_{-1} = 1$, $F_{-2} = -1$, $F_{-3} = 2$, and quite generally, $F_{-n} = (-1)^{n+1}F_n$. Accordingly, the 2h chains of Fibonacci-wise triangles extend indefinitely in both directions.

3. Again, the Fibonacci relationship

$$F_k + F_{k+1} = F_{k+2}$$

is observed to be valid for all real values of k for the added two compatible definitions

$$F_k = k \quad \text{for } 0 \le k \le 1, \quad \text{and} \quad F_k = 1 \quad \text{for } 1 \le k \le 2 .$$

Hence one obtains a non-denumerably infinite set of Fibonacci-wise chains of triangles for any prescribed positive integral value of h, wherein individual triangles of neighboring chains extend continuously along the sets of real Fibonacci numbers employed as rectangular cartesian coordinates of vertices.

★ ★ ★ ★ ★