ON SUMMATION FORMULAS AND IDENTITIES FOR FIBONACCI NUMBERS

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1. REMARKS ON THE PAPER OF BROTHER U. ALFRED

Alfred [1] has shown that

$$(1.1)\qquad \sum_{k=0}^{n-1} k^m F_{k+r} = \sum_{i=0}^{m} (-1)^i F_{2i+n+r+1} \Delta^i(n^m) + C_1 ,$$

where C_1 is a constant independent of n and $\Delta g(n) = g(n+1) - g(n)$, with $\Delta^i g(n) = \Delta(\Delta^{i-1} g(n))$. The following result yields (1.1) as a special case:

<u>Theorem 1.</u> Let $H_{n+2} = H_{n+1} + H_n$, $n = 0, 1, \cdots$, with $H_0 = \rho$ and $H_1 = \sigma$. Then for $n = 1, 2, \cdots$, we have

$$(1.2)\qquad \sum_{k=0}^{n-1} k^m H_{k+r} = H_{n+r} \sum_{s=0}^{m} \left[\binom{m}{s} \sum_{i=0}^{m} (-1)^i (i!) F_{2i} G_{m-s}^i \right] n^s$$
$$+ H_{n+r+1} \sum_{s=0}^{m} \left[\binom{m}{s} \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} G_{m-s}^i \right] n^s + C_2$$
$$(r, m = 0, 1, \cdots),$$

where

$$(1.3)\qquad C_2 = -H_r \sum_{i=0}^{m} (-1)^i (i!) F_{2i} G_m^i - H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} G_m^i$$
$$(r, m = 0, 1, \cdots),$$

and G_m^i (see [2]) are Stirling numbers of the second kind with the properties that $G_0^i = 0$ if $i \neq 0$, $G_i^i = 1$, $i = 0, 1, \cdots$, $G_i^0 = 0$ if $i \neq 0$, and $G_s^i = 0$ if $i > s$.

Proof of Theorem 1. We assert that

$$(1.4)\qquad \sum_{k=0}^{n-1} k^m H_{k+r} = \sum_{i=0}^{m} (-1)^i H_{2i+n+r+1} \Delta^i (n^m) + C_2 \quad ,$$

We note that if $\Delta g(n) = \Delta h(n)$, then $g(n) = h(n) + C_2$. Thus, using the Δ operator on both sides of (1.4), we obtain

$$(1.5)\qquad n^m H_{n+r} = \sum_{i=0}^{m} (-1)^i H_{2i+n+r+2} \Delta^i (n+1)^m - \sum_{i=0}^{m} (-1)^i H_{2i+n+r+1} \Delta^i (n^m).$$

Since $(n+1)^m - n^m = \Delta(n^m)$, we have $\Delta^i (n+1)^m = \Delta^i (n^m) + \Delta^{i+1}(n^m)$. Thus, since $H_{n+2} = H_{n+1} + H_n$, (1.5) simplifies to

$$(1.6)\qquad n^m H_{n+r} = \sum_{j=0}^{m} (-1)^j H_{2j+n+r+2} \Delta^{j+1} (n^m) + \sum_{i=0}^{m} (-1)^i H_{2i+n+r} \Delta^i (n^m).$$

Let $j + 1 = i$ in the first sum of (1.6). Since $\Delta^{m+1}(n^m) = 0$, the right-hand side sums cancel, except the term for $i = 0$, which yields $n^m H_{n+r}$.

We proceed now to simplify (1.4). Since [2, p. 9]

$$(1.7)\qquad \Delta^i g(n) = (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} g(n+k) \qquad (i = 0, 1, \cdots) \quad ,$$

we have for $g(n) = n^m$

$$(1.8) \qquad \begin{aligned}\Delta^i(n^m) &= (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} (n+k)^m \\ &= (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} \sum_{s=0}^{m} \binom{m}{s} k^{m-s} n^s \\ &= \sum_{s=0}^{m} \binom{m}{s} n^s (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} k^{m-s} \\ &= i! \sum_{s=0}^{m} \binom{m}{s} G^i_{m-s} n^s \quad ,\end{aligned}$$

since [2, p. 169, (3)]

$$(1.9) \qquad (-1)^i (i!) G^i_m = \sum_{k=0}^{i} (-1)^k \binom{i}{k} k^m \qquad (i = 0, 1, \cdots, m) \quad .$$

Buschman [3, p. 6, (12)] showed that

$$(1.10) \qquad H_{p+s} = F_s H_{p-1} + F_{s+1} H_p \quad ,$$

and thus from (1.10), with $s = 2i$ and $p = n + r + 1$, we obtain

$$(1.11) \qquad H_{2i+n+r+1} = F_{2i} H_{n+r} + F_{2i+1} H_{n+r+1} \quad .$$

Using (1.11), we obtain from (1.4)

$$(1.12) \qquad \begin{aligned}\sum_{k=0}^{n-1} k^m H_{k+r} &= H_{n+r} \sum_{i=0}^{m} (-1)^i F_{2i} \Delta^i(n^m) \\ &\quad + H_{n+r+1} \sum_{i=0}^{m} (-1)^i F_{2i+1} \Delta^i(n^m) + C_2 \quad .\end{aligned}$$

If we substitute for $\Delta^i(n^m)$ in (1.12) by (1.8), we obtain, upon interchanging summations, (1.2). Add $n^m H_{n+r}$ to both sides of (1.2). Then, for $n = 0$, all terms in the sums are 0 except for $s = 0$, and so we obtain C_2 as given by (1.3).

If $p = 0$ and $q = 1$, then $H_n \equiv F_n$, and C_2 (1.3) yields C_1 in (1.1). For calculation purposes, (1.2) is more suitable than (1.1), since Stirling numbers are tabulated. Moreover, (1.2) and (1.3) are in the simplest form possible. Using the properties of F_n and G_n^i, we note that the coefficient of H_{n+r} in (1.2) is a polynomial in n of degree $m - 1$, while the coefficient of H_{n+r+1} is a polynomial in n of degree m.

The following result is a generalization of Theorem 1:

Theorem 2. Let

$$P(x) = \sum_{j=0}^{m} a_j x^j, \qquad a_m \neq 0 \quad ,$$

where a_j, $j = 0, 1, \cdots, m$, are constants. Then for $n = 1, 2, \cdots$, we have

$$(1.13) \quad \sum_{k=0}^{n-1} P(k) H_{k+r} = H_{n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{2i} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s$$

$$+ H_{n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s + C_3$$

$$(r, m = 0, 1, \cdots),$$

where

$$(1.14) \qquad C_3 = -H_r \sum_{i=0}^{m} (-1)^i (i!) F_{2i} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$

$$- H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\}$$

$$(r, m = 0, 1, \cdots).$$

Comments. If $a_j = 0$, $j = 0, 1, \cdots, m-1$, and $a_m = 1$, then (1.13) and (1.14) reduce to (1.2) and (1.3), respectively. A special case of (1.13) occurs when

$$P(k) \equiv k^{(m)} \equiv k(k-1)\cdots(k-m+1) = \sum_{j=1}^{m} S_m^j k^j ,$$

where (see [2, p. 142]) S_m^j are Stirling numbers of the first kind. Then, since $k^{(m)} = m! \binom{k}{m}$, we have

$$\sum_{k=0}^{n-1} k^{(m)} H_{k+r} = m! \sum_{k=m}^{n-1} \binom{k}{m} H_{k+r} \qquad (n = m+1, m+2, \cdots) .$$

Moreover, since $a_j = S_m^j$, $j = 0, 1, \cdots, m$, we have

$$\sum_{j=i}^{m} a_j G_j^i = \sum_{j=i}^{m} S_m^j G_j^i = \binom{0}{m-i} = \begin{bmatrix} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{bmatrix}$$

(see [2, p. 182, (1)]). Using (1.10), we obtain from (1.14)

$$(1.15) \qquad C_3 = (-1)^{m+1}(m!)(F_{2m}H_r + F_{2m+1}H_{r+1}) = (-1)^{m+1}(m!)H_{2m+r+1} .$$

It should be noted that C_3 in (1.14) was obtained from (1.13) for $n = 0$. For $P(k) \equiv k^{(m)}$, the same value of C_3 (1.15) is also obtained from (1.13) for $n = 0, 1, \cdots, m-1$ $(m \geq 1)$. Let $P(k) \equiv k^{(m)}$ in (1.13), where $a_j = S_m^j$, and let (1.13) be written as follows:

$$(1.16) \qquad \sum_{k=0}^{n} k^{(m)} H_{k+r} - n^{(m)} H_{n+r}$$

$$= L_1(m,n)H_{n+r} + L_2(m,n)H_{n+r+1} - (-1)^m (m!)H_{2m+r+1} .$$

We obtain from (1.16)

$$(1.17)\qquad (-1)^m(m!)H_{2m+r+1} = L_1(m,n)H_{n+r} + L_2(m,n)H_{n+r+1}$$
$$(n = 0, 1, \cdots, m-1).$$

From (1.10) with $p = n + r + 1$ and $s = 2m - n$, we obtain

$$(1.18)\qquad H_{2m+r+1} = F_{2m-n}H_{n+r} + F_{2m+1-n}H_{n+r+1} .$$

If we substitute for H_{2m+r+1} in (1.17) by (1.18) and then equate coefficients of H_{n+r} and H_{n+r+1} in (1.17), we obtain the following identities:

$$(1.19)\qquad (-1)^m(m!)F_{2m-n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{2i}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$
$$(n = 0, 1, \cdots, m-1;\ m = 1, 2, \cdots),$$

$$(1.20)\qquad (-1)^m(m!)F_{2m+1-n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{2i+1}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}\right]n^s$$
$$(n = 0, 1, \cdots, m-1;\ m = 1, 2, \cdots).$$

By repeated additions, one obtains (interchanging summations in the final result)

$$(1.21)\qquad (-1)^m(m!)F_{2m+k-n} = \sum_{i=0}^{m}(-1)^i(i!)F_{2i+k}\sum_{s=0}^{m}\left\{\sum_{j=s+i}^{m}\binom{j}{s} S_m^j G_{j-s}^i\right\}n^s$$
$$(k = 0, 1, \cdots;\ n = 0, 1, \cdots, m-1;\ m = 1, 2, \cdots).$$

<u>Proof of Theorem 2.</u> Noting that $\Delta^{m+1}P(n) = 0$, we find, imitating the proof of Theorem 1, that

$$\sum_{k=0}^{n-1} P(k)H_{k+r} = \sum_{i=0}^{m} (-1)^i H_{2i+n+r+1} \Delta^i P(n) + C_3$$

$$= H_{n+r} \sum_{i=0}^{m} (-1)^i F_{2i} \Delta^i P(n) + H_{n+r+1} \sum_{i=0}^{m} (-1)^i F_{2i+1} \Delta^i P(n) + C_3$$

Since

$$P(n) = \sum_{j=0}^{m} a_j n^j, \quad \Delta^i P(n) = \sum_{j=0}^{m} a_j \Delta^i (n^j)$$

and using (1.8), we have

$$\sum_{i=0}^{m} (-1)^i F_{2i} \Delta^i P(n) = \sum_{i=0}^{m} (-1)^i F_{2i} \sum_{j=0}^{m} a_j \Delta^i (n^j)$$

$$= \sum_{i=0}^{m} (-1)^i F_{2i} \sum_{j=0}^{m} a_j (i!) \sum_{s=0}^{j} \binom{j}{s} G^i_{j-s} n^s$$

$$= \sum_{i=0}^{m} (-1)^i (i!) F_{2i} \sum_{s=0}^{m} \left\{ \sum_{j=s}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} n^s$$

$$= \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{2i} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s ,$$

since

$$\sum_{j=0}^{m} \sum_{s=0}^{j} f(s,j) = \sum_{s=0}^{m} \sum_{j=s}^{m} f(s,j) \text{ and } G^i_{j-s} = 0 \text{ if } j - s < i .$$

The value of C_3 is obtained from (1.13) for $n = 0$.

2. REMARKS ON THE PAPER BY R. REICHMAN

The operator Δ, where $\Delta g(n) = g(n+1) - g(n)$, is referred to as the forward difference operator, while the operator ∇, where $\nabla g(n) = g(n) - g(n-1)$, is referred to as the backward difference operator. Indeed,

$$(2.1) \quad \nabla^i g(n) = \sum_{s=0}^{i} (-1)^s \binom{i}{s} g(n-s) = (-1)^i \sum_{k=0}^{i} (-1)^k \binom{i}{k} g(n-i+k) .$$

If we compare (2.1) and (1.7), we note that

$$\nabla^i g(n) \equiv \Delta^i g(n - i) \qquad (i = 0, 1, \cdots) ; \tag{2.2}$$

and if $g(n) \equiv n^m$, we have

$$\nabla^i (n^m) \equiv \Delta^i (n - i)^m \qquad (i = 0, 1, \cdots, m + 1) . \tag{2.3}$$

Reichman [4] gave the following results:

$$\sum_{k=0}^{n} k^m F_k = \sum_{i=0}^{m} (-1)^i F_{n+2+i} \nabla^i (n^m) + C_4 , \tag{2.4}$$

$$\sum_{k=0}^{n} k^m F_{2k} = \sum_{i=0}^{m} (-1)^i F_{2n+1-i} \nabla^i (n^m) + C_5 , \tag{2.5}$$

$$\sum_{k=0}^{n} k^m F_{2k-1} = \sum_{i=0}^{m} (-1)^i F_{2n-i} \nabla^i (n^m) + C_6 . \tag{2.6}$$

Rao [5] generalized (2.4) and gave

$$\sum_{k=0}^{n} k^m H_k = \sum_{i=0}^{m} (-1)^i H_{n+2+i} \nabla^i (n^m) + C_4^{\star} . \tag{2.7}$$

The following results contain (2.4), (2.5), (2.6) and (2.7) as special cases. The notation is consistent with Theorems 1 and 2.

Theorem 3. For $n = 0, 1, \cdots; r = 0, \pm 1, \pm 2, \cdots,$ we have

$$(2.8)\qquad \sum_{k=0}^{n} P(k)H_{2k+r}$$

$$= -H_{2n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ H_{2n+r+1}\sum_{s=0}^{m}(-1)^s\left[(-1)^s a_s + \sum_{i=1}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_7 \qquad (m = 0, 1, \cdots),$$

where

$$(2.9)\qquad C_7 = H_r\left[a_0 + \sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right]$$

$$- H_{r+1}\left[a_0 + \sum_{i=1}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right].$$

Proof of Theorem 3. Since $\nabla P(n+1) = P(n+1) - P(n)$, we have $\nabla^i P(n) = \nabla^i P(n+1) - \nabla^{i+1}P(n+1)$, and $\nabla^{m+1}P(n+1) = 0$. Thus, imitating the proof of Theorem 1, we find that

$$\sum_{k=0}^{n} P(k)H_{2k+r} = \sum_{i=0}^{m}(-1)^i H_{2n+r+1-i}\nabla^i P(n) + C_7$$

$$= H_{2n+r}\sum_{i=0}^{m}(-1)^i F_{-i}\nabla^i P(n) + H_{2n+r+1}\sum_{i=0}^{m}(-1)^i F_{1-i}\nabla^i P(n) + C_7,$$

since $H_{2n+r+1-i} = F_{-i}H_{2n+r} + F_{1-i}H_{2n+r+1}$, which is obtained from (1.10) where $S = -i$ and $p = 2n + r + 1$. Using (2.1) and (1.9), we obtain

$$\nabla^i n^j = (-1)^{j+i}(i!)\sum_{s=0}^{j}(-1)^s\binom{j}{s}G^i_{j-s}n^s$$

Since

$$\nabla^i P(n) = \sum_{j=0}^{m} a_j \nabla^i n^j \ ,$$

we have

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i F_{-i}\nabla^i P(n) &= \sum_{i=0}^{m} i!\, F_{-i}\sum_{j=0}^{m}\sum_{s=0}^{j}(-1)^{s+j}a_j\binom{j}{s}G^i_{j-s}n^s \\ &= \sum_{i=0}^{m} i!\, F_{-i}\sum_{s=0}^{m}(-1)^s n^s\sum_{j=s}^{m}(-1)^j a_j\binom{j}{s}G^i_{j-s} \\ &= \sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\, F_{-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G^i_{j-s}\right\}\right]n^s \ .\end{aligned}$$

Additional simplifications are obtained by noting that $F_{-i} = (-1)^{i+1}F_i$ and $F_{-(i-1)} = (-1)^i F_{i-1}$. The value of C_7 is obtained from (2.8) for $n = 0$.

Comments. We note that (2.5) and (2.6) are special cases of (2.8). Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m}(-1)^j S^j_m k^j \ .$$

Since $(-k)^{(m)} = (-k)(-k-1)\cdots(-k-m+1) = (-1)^m k(k+1)\cdots(k+m-1)$, we have

$$\sum_{k=0}^{n}(-k)^{(m)}H_{2k+r} = (-1)^m(m!)\sum_{k=1}^{n}\binom{k+m-1}{m}H_{2k+r} \ ,$$

and

$$\sum_{j=i}^{m}(-1)^j a_j G_j^i = \sum_{j=i}^{m} S_m^j G_j^i = \begin{bmatrix} 1 & \text{if} & i = m \\ 0 & \text{if} & i \neq m \end{bmatrix} \quad . \tag{2.10}$$

Thus, from (2.9), with $a_0 = 0$ and $a_j = (-1)^j S_m^j$, $j = 1, \cdots, m$, we obtain (using (1.10))

$$\begin{aligned} C_7 &= (-1)^m (m!)(F_m H_r - F_{m-1} H_{r+1}) \\ &= -(m!)(F_{-m} H_r + F_{1-m} H_{r+1}) = -(m!) H_{r+1-m} \quad . \end{aligned} \tag{2.11}$$

The following result, derived via forward differences, is an alternate form of Theorem 3, which was derived via backward differences.

Theorem 4. For $n = 0, 1, \cdots; r = 0, \pm 1, \pm 2, \cdots$, we have

$$\begin{aligned} \sum_{k=0}^{n} P(k) H_{2k+r} &= H_{2n+r} \sum_{s=0}^{m} \left[\sum_{i=1}^{m} (-1)^i (i!) F_{i-2} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s \\ &+ H_{2n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!) F_{i-1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i \right\} \right] n^s + C_7 \\ & \qquad (m = 0, 1, \cdots) \, , \end{aligned} \tag{2.12}$$

where

$$C_7 = H_r \left[a_0 - \sum_{i=1}^{m} (-1)^i (i!) F_{i-2} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} \right] - H_{r+1} \sum_{i=0}^{m} (-1)^i (i!) F_{i-1} \left\{ \sum_{j=i}^{m} a_j G_j^i \right\} . \tag{2.13}$$

Comments. If we compare (2.8) with (2.12), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(2.14)\qquad (-1)^{s+1}\sum_{i=0}^{m}(-1)^i(i!)F_i\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$=\sum_{i=1}^{m}(-1)^i(i!)F_{i-2}\left\{\sum_{j=s+i}^{m}a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$(s = 0, 1, \cdots, m-1)\ ;$$

$$(2.15)\qquad (-1)^{s}\sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$=\sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$(s = 0, 1, \cdots, m)\ .$$

For $a_j = S_m^j$, $j = 0, 1, \cdots, m$, (2.14) and (2.15) with $s = 0$, yield (noting (2.10)), respectively,

$$(2.16)\qquad (-1)^{m-1}(m!)F_{m-2} = \sum_{i=0}^{m}(-1)^i(i!)F_i\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 1, 2, \cdots)\ ;$$

$$(2.17)\qquad (-1)^{m}(m!)F_{m-1} = \sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 0, 1, \cdots)\ .$$

Addition of (2.16) and (2.17) gives

$$(2.18)\qquad (-1)^{m}(m!)F_{m-3} = \sum_{i=0}^{m}(-1)^i(i!)F_{i+1}\sum_{j=i}^{m}(-1)^j S_m^j G_j^i \qquad (m = 1, 2, \cdots)\ .$$

Since $L_n = F_{n+1} + F_{n-1}$, addition of (2.17) and (2.18) gives

$$(2.19)\qquad (-1)^m(m!)L_{m-2} = \sum_{i=0}^{m}(-1)^i(i!)L_i\sum_{j=i}^{m}(-1)^jS_m^jG_j^i \qquad (m = 1,2,\cdots) .$$

We note that (2.17) may be written as

$$(2.20)\qquad (m!)F_{m-1}\left[-1+(-1)^m\right] = \sum_{i=0}^{m-1}(-1)^i(i!)F_{i-1}\sum_{j=i}^{m}(-1)^jS_m^jG_j^i$$

$$(m = 1,2,\cdots) .$$

Thus, for $m = 2n$, $n = 1,2,\cdots$, (2.20) gives

$$(2.21)\qquad \sum_{i=0}^{2n-1}(-1)^i(i!)F_{i-1}\sum_{j=i}^{2n}(-1)^jS_{2n}^jG_j^i = 0 \quad (n = 1,2,\cdots) .$$

Since ([2, pp. 149, 171])

$$S_{2n}^{2n-1} = -\binom{2n}{2} = -G_{2n}^{2n-1} ,$$

(2.21) may be written as

$$(2.22)\qquad (2n)!\,(2n-1)F_{2n-2} = \sum_{i=0}^{2n-2}(-1)^i(i!)F_{i-1}\sum_{j=i}^{2n}(-1)^jS_{2n}^jG_j^i$$

$$(n = 1,2,\cdots) .$$

Suppose now

$$P(k) \equiv k^{(m)} = \sum_{j=1}^{m}S_m^jk^j$$

in (2.12). Noting (2.10), we obtain from (2.13)

$$(2.23)\qquad C_7 = (-1)^{m+1}(m!)(F_{m-2}H_r + F_{m-1}H_{r+1}) = (-1)^{m+1}(m!)H_{r+m-1} \quad .$$

If we rewrite (2.12) as

$$(2.24)\qquad \sum_{k=0}^{n} k^{(m)}H_{2k+r} = L_1^{\star}(m,n)H_{2n+r} + L_2^{\star}(m,n)H_{2n+r+1} + C_7 \quad ,$$

we obtain from (2.24)

$$(2.25)\qquad (-1)^m(m!)H_{r+m-1} = L_1^{\star}(m,n)H_{2n+r} + L_2^{\star}(m,n)H_{2n+r+1} \quad (n = 0,1,\cdots,m-1) \;.$$

From (1.10) with $p = 2n + r + 1$ and $s = m - 2 - 2n$, we obtain

$$(2.26)\qquad H_{r+m-1} = F_{m-2-2n}H_{2n+r} + F_{m-1-2n}H_{2n+r+1} \quad .$$

If we substitute for H_{r+m-1} in (2.25) by (2.26) and then equate coefficients of H_{2n+r} and H_{2n+r+1} in (2.25), we obtain the following identities:

$$(2.27)\qquad (-1)^m(m!)F_{m-2-2n} = \sum_{s=0}^{m}\left[\sum_{i=1}^{m}(-1)^i(i!)F_{i-2}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s$$

$$(n = 0,1,\cdots,m-1;\ m = 1,2,\cdots) \quad ,$$

$$(2.28)\qquad (-1)^m(m!)F_{m-1-2n} = \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)F_{i-1}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s$$

$$(n = 0,1,\cdots,m-1;\ m = 1,2,\cdots) \;.$$

<u>Proof of Theorem 4.</u> It is readily verified that

$$\text{(2.29)}\qquad \sum_{k=0}^{n-1} P(k)H_{2k+r} = \sum_{i=0}^{m}(-1)^i H_{2n+r-1+i}\,\Delta^i P(n) + C_7$$

$$= H_{2n+r}\left[-P(n) + \sum_{i=1}^{m}(-1)^i F_{i-2}\,\Delta^i P(n)\right]$$

$$+ H_{2n+r+1}\sum_{i=0}^{m}(-1)^i F_{i-1}\,\Delta^i P(n) + C_7 \quad ,$$

since $H_{2n+r-1+i} = F_{i-2}H_{2n+r} + F_{i-1}H_{2n+r+1}$, which is obtained from (1.10) where $s = i - 2$ and $p = 2n + r + 1$. The simplification of (2.29) to the form (2.12) proceeds in the same manner as in the proof of Theorem 2. The value of C_7 (2.13) is obtained from (2.12) for $n = 0$.

The following result, derived via backward differences, is an alternate form of Theorem 2, which was derived via forward differences. Since

$$\text{(2.30)}\qquad \sum_{k=0}^{n} P(k)H_{k+r} = \sum_{i=0}^{m}(-1)^i H_{n+r+2+i}\,\nabla^i P(n) + C_3$$

$$= H_{n+r}\left[P(n) + \sum_{i=1}^{m}(-1)^i F_{i+1}\,\nabla^i P(n)\right]$$

$$+ H_{n+r+1}\sum_{i=0}^{m}(-1)^i F_{i+2}\,\nabla^i P(n) + C_3 \quad ,$$

we may now state

<u>Theorem 5.</u> For $m = 0, 1, \cdots; n = 1, 2, \cdots$,

$$\text{(2.31)}\qquad \sum_{k=0}^{n-1} P(k)H_{k+r} = H_{n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=1}^{m} i!\,F_{i+1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ H_{n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,F_{i+2}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_3 \quad ,$$

where

$$(2.32)\qquad C_3 = -H_r \sum_{i=1}^{m} i!\, F_{i+1}\left\{\sum_{j=i}^{m} (-1)^j a_j G_j^i\right\} - H_{r+1}\sum_{i=0}^{m} i!\, F_{i+2}\left\{\sum_{j=i}^{m} (-1)^j a_j G_j^i\right\}$$

$$(r, m = 0, 1, \cdots) \quad .$$

Comments. If we compare (1.13) with (2.31), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(2.33)\qquad \sum_{i=1}^{m} (-1)^i (i!) F_{2i}\left\{\sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i\right\}$$
$$= (-1)^s \sum_{i=1}^{m} i!\, F_{i+1}\left\{\sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^i\right\}$$
$$(s = 0, 1, \cdots, m-1) \quad ;$$

$$(2.34)\qquad \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1}\left\{\sum_{j=s+i}^{m} a_j \binom{j}{s} G_{j-s}^i\right\}$$
$$= (-1)^s \sum_{i=0}^{m} i!\, F_{i+2}\left\{\sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G_{j-s}^i\right\}$$
$$(s = 0, 1, \cdots, m) \quad .$$

For $a_j = (-1)^j S_m^j$, $j = 0, 1, \cdots, m$, (2.33) and (2.34) with $s = 0$, yield (noting (2.10)), respectively

$$(2.35)\qquad m!\, F_{m+1} = \sum_{i=1}^{m} (-1)^i (i!) F_{2i} \sum_{j=i}^{m} (-1)^j S_m^j G_j^i$$

$$(m = 1, 2, \cdots) \quad ;$$

(2.36) $$m!\,F_{m+2} = \sum_{i=0}^{m} (-1)^i (i!) F_{2i+1} \sum_{j=i}^{m} (-1)^j S_m^j G_j^i \qquad (m = 0, 1, \cdots) .$$

Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m} (-1)^j S_m^j k^j$$

in (2.31). Then

$$\sum_{k=0}^{n-1} (-k)^{(m)} H_{k+r} = (-1)^m (m!) \sum_{k=1}^{n-1} \binom{k+m-1}{m} H_{k+r} ,$$

and from (2.32) we obtain

(2.37) $$C_3 = -(m!)(F_{m+1}H_r + F_{m+2}H_{r+1}) = -(m!)H_{m+r+2} .$$

We note that (2.4) and (2.7) are special cases of (2.30).

3. ADDITIONAL RESULTS

In terms of forward differences it is readily verified that

(3.1) $$\sum_{k=0}^{n-1} P(k) H_{3k+r} = \sum_{i=0}^{m} (-1)^i 2^{-i-1} H_{3n+r-1+2i}\, \Delta^i P(n) + C_8$$

$$= H_{3n+r} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2i-2}\, \Delta^i P(n)$$

$$+ H_{3n+r+1} \sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2i-1}\, \Delta^i P(n) + C_8 .$$

Moreover, in terms of backward differences, it is readily verified that

$$(3.2)\qquad \sum_{k=0}^{n} P(k)H_{3k+r} = \sum_{i=0}^{m} (-1)^i 2^{-i-1} H_{3n+r+2-i}\nabla^i P(n) + C_8$$

$$= H_{3n+r}\sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{1-i}\nabla^i P(n)$$

$$+ H_{3n+r+1}\sum_{i=0}^{m} (-1)^i 2^{-i-1} F_{2-i}\nabla^i P(n) + C_8 \ .$$

The following result is a restatement of (3.1) and (3.2):

Theorem 6. For $n = 1, 2, \cdots$; $r = 0, \pm 1, \pm 2, \cdots$, we have

$$(3.3)\qquad \sum_{k=0}^{n-1} P(k)H_{3k+r} = H_{3n+r}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ H_{3n+r+1}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_8 \qquad\qquad (m = 0, 1, \cdots) \quad ,$$

where

$$(3.4)\qquad C_8 = -H_r\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=i}^{m} a_jG_j^i\right\}$$

$$- H_{r+1}\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=i}^{m} a_jG_j^i\right\} \quad .$$

For $n = 0, 1, \cdots$; $r = 0, \pm 1, \pm 2, \cdots$, we have

$$(3.5)\qquad \sum_{k=0}^{n} P(k)H_{3k+r} = H_{3n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+\ H_{3n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+\ C_8 \qquad (m = 0, 1, \cdots)\quad ,$$

where

$$(3.6)\qquad C_8 = H_r\left[a_0 - \sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right]$$

$$-\ H_{r+1}\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\quad .$$

Comments. Add $P(n)H_{3n+r}$ to both sides of (3.3). Then, comparing (3.3) and (3.5), we conclude that for arbitrary a_j, $j = 0, 1, \cdots, m$,

$$(3.7)\qquad a_s + \sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$= (-1)^s\sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$(s = 0, 1, \cdots, m)\quad ;$$

$$(3.8)\qquad \sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}$$

$$= (-1)^s\sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\qquad (s = 0, 1, \cdots, m)\quad .$$

For $a_j = S_m^j$, $j = 0, 1, \cdots, m$, (3.7) and (3.8) with $s = 0$, yield (noting (2.10)), respectively

$$(3.9)\qquad m!\,2^{-m-1}F_{2m-2} = \sum_{i=0}^{m} i!\,2^{-i-1}F_{1-i}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\}\quad (m = 1, 2, \cdots),$$

$$(3.10)\qquad m!\,2^{-m-1}F_{2m-1} = \sum_{i=0}^{m} i!\,2^{-i-1}F_{2-i}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\}\quad (m = 0, 1, \cdots),$$

which may be simplified by noting that $F_{1-i} = (-1)^i F_{i-1}$ and $F_{2-i} = (-1)^{i+1}F_{i-2}$.

If $a_j = (-1)^j S_m^j$, $j = 0, 1, \cdots, m$, (3.7) and (3.8) with $s = 0$ yield, respectively,

$$(3.11)\qquad (m!)2^{-m-1}F_{m-1} = \sum_{i=0}^{m}(-1)^i (i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\}\qquad (m = 1, 2, \cdots),$$

$$(3.12)\qquad -(m!)2^{-m-1}F_{m-2} = \sum_{i=0}^{m}(-1)^i (i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=i}^{m}(-1)^j S_m^j G_j^i\right\}\qquad (m = 0, 1, \cdots).$$

By repeated additions, (3.9) and (3.10), as well as (3.11) and (3.12), give similar identities for Lucas numbers, L_n.

Suppose now

$$P(k) \equiv k^{(m)} = \sum_{j=1}^{m} S_m^j k^j$$

in (3.3). Then, from (3.4), we obtain

$$\begin{aligned} C_8 &= (-1)^{m+1}(m!)2^{-m-1}(F_{2m-2}H_r + F_{2m-1}H_{r+1}) \\ &= (-1)^{m+1}2^{-m-1}H_{2m+r-1}(m!) . \end{aligned}$$

From (1.10) with $p = 3n + r + 1$ and $s = 2m - 2 - 3n$, we obtain

$$H_{2m+r-1} = F_{2m-2-3n}H_{3n+r} + F_{2m-1-3n}H_{3n+r+1} .$$

If we substitute for C_8 in (3.3) and then equate coefficients of H_{3n+r} and H_{3n+r+1}, we obtain the following identities:

$$\begin{aligned} (3.13) \quad & (-1)^m(m!)2^{-m-1}F_{2m-2-3n} \\ &= \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-2}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s \end{aligned}$$

$(n = 0, 1, \cdots, m - 1;\ m = 1, 2, \cdots)$,

$$\begin{aligned} (3.14) \quad & (-1)^m(m!)2^{-m-1}F_{2m-1-3n} \\ &= \sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)2^{-i-1}F_{2i-1}\left\{\sum_{j=s+i}^{m}\binom{j}{s}S_m^jG_{j-s}^i\right\}\right]n^s \end{aligned}$$

$(n = 0, 1, \cdots, m - 1; m = 1, 2, \cdots)$.

Suppose now

$$P(k) \equiv (-k)^{(m)} = \sum_{j=1}^{m}(-1)^jS_m^jk^j$$

in (3.5). Then from (3.6), we obtain

$$C_8 = -(m!)2^{-m-1}(F_{1-m}H_r + F_{2-m}H_{r+1}) = -(m!)2^{-m-1}H_{r+2-m} .$$

4. GENERALIZATIONS

Let a, b, U_0 and U_1 be arbitrary real numbers, and consider the following three sequences:

$$(4.1) \qquad U_{n+2} = aU_{n+1} + bU_n , \quad ab = 1, \quad a \neq -1, \quad (n = 0, 1, \cdots) ,$$

$$(4.2) \qquad U_{n+2} = aU_{n+1} + U_n , \quad a \neq 0, \quad (n = 0, 1, \cdots) ,$$

$$(4.3) \qquad U_{n+2} = U_{n+1} + bU_n , \quad b = 0, \quad (n = 0, 1, \cdots) , .$$

We note that (4.1), (4.2), and (4.3) reduce to the Fibonacci sequence for the proper choices of a and b. We shall obtain summation formulas, using both forward and backward differences, for each of the three sequences, as defined by (4.1), (4.2), and (4.3), which yield the previous results, i.e., Theorems 2, 3, 4, 5, and 6, as special cases for the proper choices of a and b. We have already seen how certain procedures may be used to obtain various identities from our Theorems 2, ⋯, 6. In view of space limitations, no attempt will be made to use these procedures to fully exploit the general results obtained in this section. Identities given in the proofs of Theorems 2 and 3 will be used to obtain the explicit formulas cited in our general theorems, whose proofs are similar to that used for Theorem 2 (if forward differences are involved) or to that used for Theorem 3 (if backward differences are involved). We shall use repeatedly the following identity [3, p. 6, 12]

$$(4.4) \qquad U_{p+s} = b\phi_s U_{p-1} + \phi_{s+1}U_p$$

where $\phi_0 = 0$, $\phi_1 = 1$, and $\phi_{n+2} = a\phi_{n+1} + b\phi_n$, $n = 0, 1, \cdots$. We note that (4.4) yields (1.10) for $a = b = 1$. All results in this section are valid for the parameter range, $r = 0, \pm 1, \pm 2, \cdots$. $P(k)$ (see Theorem 2) is defined as before. For negative subscripts, we define

$$(4.5) \qquad U_{-n} = (U_0V_n - U_n)/(-b)^n \qquad (n = 1, 2, \cdots) \ ,$$

where $V_0 = 2$, $V_1 = a$, and $V_{n+2} = aV_{n+1} + bV_n$, $n = 0, 1, \cdots$. We note that $\phi_{-n} = -\phi_n/(-b)^n$, $n = 1, 2, \cdots$.

(i) Let U_n satisfy (4.1). Since

$$(4.6) \qquad \begin{aligned} \sum_{k=0}^{n-1} P(k)U_{3k+r} &= \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} U_{3n+r-1+2i}\, \Delta^i P(n) + C_8^* \\ &= bU_{3n+r} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2i-2}\, \Delta^i P(n) \\ &\quad + U_{3n+r+1} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2i-1}\, \Delta^i P(n) + C_8^* \end{aligned}$$

and

$$(4.7) \qquad \begin{aligned} \sum_{k=0}^{n} P(k)U_{3k+r} &= \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} U_{3n+r+2-i}\, \nabla^i P(n) + C_8^* \\ &= bU_{3n+r} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{1-i}\, \nabla^i P(n) \\ &\quad + U_{3n+r+1} \sum_{i=0}^{m} (-1)^i (a^2 + b)^{-i-1} \phi_{2-i}\, \nabla^i P(n) + C_8^* \ , \end{aligned}$$

We may now state

Theorem 7. Let U_n satisfy (4.1). For $n = 1, 2, \cdots$, and $m = 0, 1, \cdots$, we have

$$(4.8)\qquad \sum_{k=0}^{n-1} P(k) U_{3k+r}$$

$$= bU_{3n+r} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-2} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s$$

$$+ U_{3n+r+1} \sum_{s=0}^{m} \left[\sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-1} \left\{ \sum_{j=s+i}^{m} a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s + C_8^\star ,$$

where

$$(4.9)\qquad C_8^\star = -bU_r \sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-2} \left\{ \sum_{j=i}^{m} a_j G^i_j \right\}$$

$$- U_{r+1} \sum_{i=0}^{m} (-1)^i (i!)(a^2 + b)^{-i-1} \phi_{2i-1} \left\{ \sum_{j=i}^{m} a_j G^i_j \right\} .$$

For $n = 0, 1, \cdots$, and $m = 0, 1, \cdots$, we have

$$(4.10)\qquad \sum_{k=0}^{n} P(k) U_{3k+r}$$

$$= bU_{3n+r} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\,(a^2 + b)^{-i-1} \phi_{1-i} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s$$

$$+ U_{3n+r+1} \sum_{s=0}^{m} (-1)^s \left[\sum_{i=0}^{m} i!\,(a^2 + b)^{-i-1} \phi_{2-i} \left\{ \sum_{j=s+i}^{m} (-1)^j a_j \binom{j}{s} G^i_{j-s} \right\} \right] n^s + C_8^\star ,$$

where

$$(4.11)\qquad C_8^\star = U_r\left[a_0 - b\sum_{i=0}^{m} i!\,(a^2+b)^{-i-1}\phi_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right]$$
$$- U_{r+1}\sum_{i=0}^{m} i!\,(a^2+b)^{-i-1}\phi_{2-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\} .$$

For $a = b = 1$, Theorem 7 yields Theorem 6.

(ii) Let U_n satisfy (4.2). Since

$$(4.12)\qquad \sum_{k=0}^{n-1} P(k)U_{2k+r} = \sum_{i=0}^{m}(-1)^i a^{-i-1}U_{2n+r-1+i}\,\Delta^i P(n) + C_7^\star$$
$$= U_{2n+r}\left[-P(n) + \sum_{i=1}^{m}(-1)^i a^{-i-1}\phi_{i-2}\,\Delta^i P(n)\right]$$
$$+ U_{2n+r+1}\sum_{i=0}^{m}(-1)^i a^{-i-1}\phi_{i-1}\,\Delta^i P(n) + C_7^\star$$

and

$$(4.13)\qquad \sum_{k=0}^{n} P(k)U_{2k+r} = \sum_{i=0}^{m}(-1)^i a^{-i-1}U_{2n+r+1-i}\,\nabla^i P(n) + C_7^\star$$
$$= U_{2n+r}\sum_{i=0}^{m}(-1)^i a^{-i-1}\phi_{-i}\,\nabla^i P(n)$$
$$+ U_{2n+r+1}\sum_{i=0}^{m}(-1)^i a^{-i-1}\phi_{1-i}\,\nabla^i P(n) + C_7^\star ,$$

we may now state

Theorem 8. Let U_n satisfy (4.2). For $n, m = 0, 1, \cdots,$ we have

$$(4.14)\quad \sum_{k=0}^{n} P(k)U_{2k+r}$$

$$= U_{2n+r}\sum_{s=0}^{m}\left[\sum_{i=1}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-2}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ U_{2n+r+1}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s + C_7^\star ,$$

where

$$(4.15)\quad C_7^\star = U_r\left[a_0 - \sum_{i=1}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-2}\sum_{j=i}^{m}a_jG_j^i\right]$$

$$- U_{r+1}\sum_{i=0}^{m}(-1)^i(i!)a^{-i-1}\phi_{i-1}\sum_{j=i}^{m}a_jG_j^i .$$

For $n, m = 0, 1, \cdots,$ we have

$$(4.16)\quad \sum_{k=0}^{n} P(k)U_{2k+r}$$

$$= U_{2n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\, a^{-i-1}\phi_{-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ U_{2n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\, a^{-i-1}\phi_{1-i}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s}G_{j-s}^{i}\right\}\right]n^s$$

$$+ C_7^\star ,$$

where

$$(4.17)\qquad C_7^\star = U_r\left[a_0 - \sum_{i=0}^{m} i!\, a^{-i-1}\phi_{-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}\right] - U_{r+1}\sum_{i=0}^{m} i!\, a^{-i-1}\phi_{1-i}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\} .$$

For $a = 1$, (4.14) and (4.15) yield Theorem 4; and (4.16) and (4.17) yield Theorem 3.

(iii) Let U_n satisfy (4.3). Since

$$(4.18)\qquad \sum_{k=0}^{n-1} P(k)U_{k+r} = \sum_{i=0}^{m}(-1)^i b^{-i-1}U_{n+r+1+2i}\,\Delta^i P(n) + C_3^\star$$
$$= bU_{n+r}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{2i}\,\Delta^i P(n) + U_{n+r+1}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{2i+1}\,\Delta^i P(n) + C_3^\star$$

and

$$(4.19)\qquad \sum_{k=0}^{n} P(k)U_{k+r} = \sum_{i=0}^{m}(-1)^i b^{-i-1}U_{n+r+2+i}\nabla^i P(n) + C_3^\star$$
$$= bU_{n+r}\left[b^{-1}P(n) + \sum_{i=1}^{m}(-1)^i b^{-i-1}\phi_{i+1}\nabla^i P(n)\right] + U_{n+r+1}\sum_{i=0}^{m}(-1)^i b^{-i-1}\phi_{i+2}\,\nabla^i P(n) + C_3^\star ,$$

We may now state

Theorem 9. Let U_n satisfy (4.3). For $m = 0, 1, \cdots$; $n = 1, 2, \cdots$, we have

$$\sum_{k=0}^{n-1} P(k)U_{k+r} \tag{4.20}$$

$$= bU_{n+r}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)b^{-i-1}\phi_{2i}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s} G_{j-s}^i\right\}\right]n^s$$

$$+ U_{n+r+1}\sum_{s=0}^{m}\left[\sum_{i=0}^{m}(-1)^i(i!)b^{-i-1}\phi_{2i+1}\left\{\sum_{j=s+i}^{m} a_j\binom{j}{s} G_{j-s}^i\right\}\right]n^s + C_3^\star ,$$

where

$$C_3^\star = -bU_r\sum_{i=0}^{m}(-1)^i(i!)b^{-i-1}\phi_{2i}\left\{\sum_{j=i}^{m} a_jG_j^i\right\} \tag{4.21}$$

$$- U_{r+1}\sum_{i=0}^{m}(-1)^i(i!)b^{-i-1}\phi_{2i+1}\left\{\sum_{j=i}^{m} a_jG_j^i\right\} .$$

For $m = 0, 1, \cdots$; $n = 1, 2, \cdots$, we have

$$\sum_{k=0}^{n-1} P(k)U_{k+r} = bU_{n+r}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=1}^{m} i!\, b^{-i-1}\phi_{i+1}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s} G_{j-s}^i\right\}\right]r \tag{4.22}$$

$$+ U_{n+r+1}\sum_{s=0}^{m}(-1)^s\left[\sum_{i=0}^{m} i!\, b^{-i-1}\phi_{i+2}\left\{\sum_{j=s+i}^{m}(-1)^j a_j\binom{j}{s} G_{j-s}^i\right.\right.$$

$$+ C_3^\star \quad ,$$

where

$$(4.23)\qquad C_3^{\star} = -bU_r\sum_{i=1}^{m} i!\,b^{-i-1}\phi_{i+1}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}$$

$$- U_{r+1}\sum_{i=0}^{m} i!\,b^{-i-1}\phi_{i+2}\left\{\sum_{j=i}^{m}(-1)^j a_j G_j^i\right\}.$$

For $b = 1$, (4.20) and (4.21) yield Theorem 2; and (4.22) and (4.23) yield Theorem 5.

5. APPLICATIONS FOR A SUMMATION FORMULA

Recently, the author [6] proved the following result:

Lemma 1. Let u_i, $i = 0, 1, \cdots, p - 1$, be arbitrary real numbers, and let u_n, $n = 0, 1, \cdots$, satisfy a homogeneous, linear difference equation of order p with real, constant coefficients.

$$(5.1)\qquad b_0u_{n+p} + b_1u_{n+p-1} + \cdots + b_pu_n = 0 \qquad (b_0b_p \neq 0)\ .$$

Let x be a real number. Then

$$(5.2)\qquad -\left[\sum_{i=0}^{p} b_i x^i\right]\sum_{k=0}^{n} u_k x^k = \sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{n+1+k-j}\right]x^{n+1+k}$$

$$- \sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{k-j}\right]x^k\ ;$$

(5.3)
$$\sum_{k=0}^{\infty} u_k x^k = \frac{\sum_{k=0}^{p-1}\left[\sum_{j=0}^{k} b_j u_{k-j}\right] x^k}{\sum_{i=0}^{p} b_i x^i}$$

The series in (5.3) converges for $|x| < |\lambda|$, where λ is the root of $b_p x^p + \cdots + b_1 x + b_0 = 0$ with the smallest absolute value.

In [6], (5.2) was used to obtain a closed form for

$$\sum_{k=0}^{n} k^p x^k \quad .$$

If x_0 is a value of x such that

$$\sum_{i=0}^{p} b_i x_0^i = 0 \quad ,$$

then

$$\sum_{k=0}^{n} u_k x_0^k$$

is obtained from (5.2) by applying L'Hospital's rule.

As before, let

$$P(k) = \sum_{j=0}^{m} a_j k^j \; , \qquad a_m \neq 0 \quad ,$$

and consider $u_k \equiv P(k) w_{qk+r}$, $k = 0, 1, \cdots$, where $q = 1, 2, \cdots$; $r = 0, \pm 1, \pm 2, \cdots$, and

$$w_{n+2} + d_1 w_{n+1} + d_2 w_n = 0, \quad d_1 d_2 \neq 0, \quad d_1^2 - 4d_2 \neq 0, \quad (n = 0, 1, \cdots). \tag{5.4}$$

If α and β are the roots of $x^2 + d_1 x + d_2 = 0$, then $U_k \equiv w_{qk+r}$ satisfies

$$U_{k+2} - V_q U_{k+1} + d_2^q U_k = 0 \quad (k = 0, 1, \cdots) , \tag{5.5}$$

since $(x - \alpha^q)(x - \beta^q) = x^2 - V_q x + d_2^q$, where $V_n = \alpha^n + \beta^n$, $n = 0, 1, \cdots$, with $V_0 = 2$, $V_1 = -d_1$, satisfies (5.4). We note that $P(k) w_{qk+r}$ is a solution of a homogeneous, linear difference equation of order $2m + 2$ with real, constant coefficients whose characteristic equation is given by

$$\left[(x - \alpha^q)(x - \beta^q)\right]^{m+1} \equiv (x^2 - V_q x + d_2^q)^{m+1} = 0 . \tag{5.6}$$

Since

$$(x^2 - V_q x + d_2^q)^{m+1} = \sum_{s=0}^{2m+2} b_{2m+2-s} x^s ,$$

we have that

$$(1 - V_q x + d_2^q x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j x^j .$$

In [2, p. 30, example 3], it is shown that

$$b_j = (-1)^j \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{i}{j-i} V_q^{2i-j} d_2^{q(j-i)} \quad (j = 0, 1, \cdots, 2m+2). \tag{5.7}$$

Thus, (5.2), in which $p = 2m + 2$ and b_j defined by (5.7), yields a closed form for

$$\sum_{k=0}^{n} P(k) w_{qk+r} x^k .$$

If $w_k \equiv H_k$, then $d_1 = d_2 = -1$, $V_q \equiv L_q$, and (5.2) yields

$$-(1 - L_q x + (-1)^q x^2)^{m+1} \sum_{k=0}^{n} P(k) H_{qk+r} x^k \tag{5.8}$$
$$= \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(n + 1 + k - j) H_{q(n+1+k-j)+r} \right] x^{n+1+k}$$
$$- \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(k - j) H_{q(k-j)+r} \right] x^k \qquad (n = 0, 1, \cdots) ,$$

where (see (5.7))

$$b_j = (-1)^{j(q+1)} \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \binom{i}{j-i} L_q^{2i-j} \quad (j = 0, 1, \cdots, 2m + 2). \tag{5.9}$$

If $P(k) = k^{(m)} = m!\binom{k}{m}$ in (5.8), we conclude that for arbitrary x,

$$\sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j \binom{n + 1 + k - j}{m} H_{q(n+1+k-j)+r} \right] x^{n+1+k} \tag{5.10}$$
$$= \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j \binom{k - j}{m} H_{q(k-j)+r} \right] x^k$$
$$(n = 0, 1, \cdots, m - 1;\ m = 1, 2, \cdots) .$$

If $n = 0$ in (5.10), the coefficient of x^{2m+2} must be 0, i.e.,

$$\sum_{j=0}^{2m+1} b_j \binom{2m + 2 - j}{m} H_{q(2m+2-j)+r} = 0 \quad (m = 1, 2, \cdots) . \tag{5.11}$$

If $P(k) = (-k)^{(m)} = (-1)^m (m!)\binom{k+m-1}{m}$ in (5.8), we conclude that for arbitrary x and $n = 0$,

$$\text{(5.12)} \quad \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j \binom{k-j+m}{m} H_{q(1+k-j)+r}\right] x^{1+k}$$

$$= \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j \binom{k-j+m-1}{m} H_{q(k-j)+r}\right] x^{k} \qquad (m = 1, 2, \cdots) .$$

In (5.12), the coefficient of x^{2m+2} must be 0, i.e.,

$$\text{(5.13)} \quad \sum_{j=0}^{2m+1} b_j \binom{3m+1-j}{m} H_{q(2m+2-j)+r} = 0 \qquad (m = 1, 2, \cdots) .$$

If $P(k) \equiv 1$, then (5.8) yields a result which has already been proved by the author [7, p. 105, (5)], using a different procedure.

Noting that $w_m = \cos m\theta$ and $w_n = \sin n\theta$ satisfy $w_{n+2} - 2\cos\theta w_{n+1} + w_n = 0$, $n = 0, 1, \cdots$, with $V_n = 2\cos n\theta$, where $\theta \neq 0, \pi$, $0 < \theta < 2\pi$, we obtain from (5.2) the following two identities:

$$\text{(5.14)} \quad -[1 - 2(\cos q\theta)x + x^2]^{m+1} \sum_{k=0}^{n} P(k) \begin{Bmatrix} \cos\ (qk+r)\theta \\ \sin\ (qk+r)\theta \end{Bmatrix} x^k$$

$$= \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j P(n+1+k-j) \begin{Bmatrix} \cos\ [q(n+1+k-j)+r]\theta \\ \sin\ [q(n+1+k-j)+r]\theta \end{Bmatrix}\right] x^{n+1+k}$$

$$- \sum_{k=0}^{2m+1}\left[\sum_{j=0}^{k} b_j P(k-j) \begin{Bmatrix} \cos\ [q(k-j)+r]\theta \\ \sin\ [q(k-j)+r]\theta \end{Bmatrix}\right] x^{k} \qquad (n = 0, 1, \cdots) ,$$

where (see (5.7))

$$\text{(5.15)} \quad b_j = (-1)^j \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{i}{j-i} (2\cos q\theta)^{2i-j} \qquad (j = 0, 1, \cdots, 2m+2).$$

The relative simplicity of our results, (5.14) and (5.15), may be compared with the less general (as well as less elegant) results obtained by Schwatt [8, pp. 217-219], who used the differential operator, $(xd/dx)^m$.

For choices of $P(k) \equiv k^{(m)}$ or $(-k)^{(m)}$, we obtain (in the same manner as (5.11) and (5.13)) the identities (pairwise)

$$(5.16)\qquad \sum_{j=0}^{2m+1} b_j \binom{2m+2-j}{m} \begin{Bmatrix} \cos\left[q(2m+2-j)+r\right]\theta \\ \sin\left[q(2m+2-j)+r\right]\theta \end{Bmatrix} = 0 \quad (m = 1, 2, \cdots),$$

$$(5.17)\qquad \sum_{j=0}^{2m+1} b_j \binom{3m+1-j}{m} \begin{Bmatrix} \cos\left[q(2m+2-j)+r\right]\theta \\ \sin\left[q(2m+2-j)+r\right]\theta \end{Bmatrix} = 0 \quad (m = 1, 2, \cdots).$$

Identities (5.16) and (5.17) may be transformed to hold for hyperbolic functions by recalling that $\cosh(i\theta) = \cos\theta$ and $\sinh(i\theta) = i\sin\theta$.

As an application of (5.3), we have

$$(5.18)\qquad (1 - V_q x + d_2^q x^2)^{m+1} \sum_{k=0}^{\infty} P(k) w_{qk+r} x^k = \sum_{k=0}^{2m+1} \left[\sum_{j=0}^{k} b_j P(k-j) w_{q(k-j)+r} \right] x^k ,$$

where b_j is defined by (5.7).

It is desirable to have check formulas for the computed values of b_j. In our discussion, consider b_j, as given by (5.7), where

$$(5.19)\qquad (1 - V_q x + d_2^q x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j x^j \quad (m = 0, 1, \cdots).$$

We may set $x = \pm 1$ in (5.19). A substantial reduction in the effort required to evaluate all the b_j, $j = 0, 1, \cdots, 2m+2$, is afforded by noting that

(5.20) $$b_{2m+2-j} = d_2^{q(m+1-j)} b_j \qquad (j = 0, 1, \cdots, m+1) .$$

To prove (5.20), multiply both sides of (5.19) by $d_2^{q(m+1)}$, and so

(5.21) $$(d_2^q - d_2 V_q x + d_2^{2q} x^2)^{m+1} = \sum_{j=0}^{2m+2} b_j d_2^{q(m+1)} x^j .$$

Replacing x in (5.21) by x/d_2^q, we obtain (in reverse order)

(5.22) $$(x^2 - V_q x + d_2^q)^{m+1} = \sum_{j=0}^{2m+2} b_j d_2^{q(m+1-j)} x^j = \sum_{j=0}^{2m+2} b_{2m+2-j} x^j ;$$

and thus (5.20) is obtained by comparing the coefficients of x^j in the sums in (5.22).

Let $t = 1, 2, \cdots$, and let $g_{t+1}(x) = 0$ (where $g_{t+1}(x)$ is a polynomial in x of degree $t+1$) be the characteristic equation determined by H_{qk+r}^t. Then the characteristic equation determined by $u_k \equiv P(k) H_{qk+r}^t$ is given by $[g_{t+1}(x)]^{m+1} = 0$. Since

$$[x^{t+1} g_{t+1}(1/x)]^{m+1} = \sum_{j=0}^{(t+1)(m+1)} b_j x^j ,$$

(5.2) may be applied to yield a closed form for

$$\sum_{k=0}^{n} P(k) H_{qk+r}^t x^k .$$

A formidable obstacle in this procedure is the complex nature of the b_j, which involve multiple summations.

As a simple example, consider H^2_{n+r}, where $H^2_{n+3+r} - 2H^2_{n+2+r} - 2H^2_{n+1+r} + H^2_{n+r} = 0$, and $g_3(x) \equiv x^3 - 2x^2 - 2x + 1$. Then $x^3 g(1/x) \equiv 1 - 2x - 2x^2 + x^3$ and

$$(1 - 2x - 2x^2 + x^3)^{m+1} \equiv \sum_{i=0}^{3(m+1)} b_j x^j \;.$$

Using the binomial theorem and then applying (5.7) (with the proper change of notation for the coefficients), we obtain

$$\begin{aligned}(1 - 2x - 2x^2 + x^3)^{m+1} &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2x)^i \left[1 + x - (x^2/2)\right]^i \\ &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2)^i \sum_{k=0}^{2i} c_k x^{k+i} = \sum_{j=0}^{3m+3} b_j x^j\end{aligned}$$

where

$$c_k = \sum_{s=0}^{i} \binom{i}{s} \binom{s}{k-s} (-1/2)^{k-s} \qquad (k = 0, 1, \cdots, 2i) \;,$$

and

$$\begin{aligned}(5.23)\qquad b_j &= \sum_{i=0}^{m+1} \binom{m+1}{i} (-2)^i c_{j-i} \\ &= (-2)^{-j} \sum_{i=0}^{m+1} 2^{2i} \binom{m+1}{i} \sum_{s=0}^{i} (-2)^s \binom{i}{s} \binom{s}{j-i-s} \\ &\qquad\qquad (j = 0, 1, \cdots, 3m+3) \;.\end{aligned}$$

Thus, from (5.2) with $p = 3m + 3$ and $u_k = P(k)H^2_{k+r}$, we obtain (where b_j is defined by (5.23)),

$$(5.24) \qquad -(1 - 2x - 2x^2 + x^3)^{m+1} \sum_{k=0}^{n} P(k) H_{k+r}^2 x^k$$

$$= \sum_{k=0}^{3m+2} \left[\sum_{j=0}^{k} b_j P(n + 1 + k - j) H_{n+1+k-j+r}^2 \right] x^{n+1+k} - \sum_{k=0}^{3m+2} \left[\sum_{j=0}^{k} b_j P(k - j) H_{k-j+r}^2 \right] x^k .$$

Recalling the manner by which (5.11), (5.13), (5.16), and (5.17) were obtained, we may now state the following result:

Theorem 10. Let

$$(5.25) \qquad [x^{t+1} g_{t+1}(1/x)]^{m+1} = \sum_{j=0}^{(t+1)(m+1)} b_j x^j \qquad (m = 1, 2, \cdots) .$$

Then

$$(5.26) \qquad \sum_{j=0}^{(t+1)(m+1)-1} b_j \binom{(t + 1)(m + 1) - j}{m} H_{q(tm+t+m+1-j)+r}^t = 0$$

$$(q, t, m = 1, 2, \cdots; \; r = 0, \pm 1, \pm 2, \cdots) ;$$

$$(5.27) \qquad \sum_{j=0}^{(t+1)(m+1)-1} b_j \binom{(t + 1)(m + 1) - j - 1 + m}{m} H_{q(tm+t+m+1-j)+r}^t = 0$$

$$(q, t, m = 1, 2, \cdots; \; r = 0, \pm 1, \pm 2, \cdots) .$$

We note that (5.26) and (5.27) are identical for $m = 1$.

6. REMARKS ON THE PAPER BY LEDIN [9]

From our (2.31) with $r = 0$, $H_k = F_k$, and $P(k) = k^m$ (so that $a_m = 1$, $a_j = 0$, $j = 0, 1, \cdots, m-1$), we conclude (see 9, (3a), (3b) for notation) that

$$M_{1,j} = \sum_{k=0}^{j} k!\, F_{k+1} G_j^k \qquad (j = 0, 1, \cdots) \tag{6.1}$$

$$M_{2,j} = \sum_{k=0}^{j} k!\, F_{k+2} G_j^k \qquad (j = 0, 1, \cdots) \tag{6.2}$$

From [9, (6a)], we obtain for $i = 3$

$$M_{3,j} = \sum_{k=0}^{j} k!\, F_{k+3} G_j^k - 0^j \qquad (j = 0, 1, \cdots) \tag{6.3}$$

Thus, the assertion [9, (6e)] is valid only for $i = 1,$) (with $j = 0, 1, \cdots$) and $i = 3$ ($j = 1, 2, \cdots$). Since $F_{k+i} = F_{i-1}F_{k+2} + F_{i-2}F_{k+1}$ (see (1.10)), we obtain from [9, (6b)], using (6.1) and (6.2) above, that

$$M_{i,j} = \sum_{k=0}^{j} k!\, F_{k+i} G_j^k - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k} \qquad (j = 1, 2, \cdots) \tag{6.4}$$

Noting (6.1), (6.2), and (6.4), we are tempted to define

$$M_{0,j} = \sum_{k=0}^{j} k!\, F_k G_j^k \qquad (j = 0, 1, \cdots) .$$

It should be noted that (6.1) and (6.2) are not uniquely defined. In the notation of [9, (8)], our (1.2) (with $r = 0$ and $H_k \equiv F_k$) can be written as

$$S(m, n-1) = F_n P_3(m,n) + F_{n-1} P_2(m,n) + C(m) \quad , \tag{6.5}$$

where (using 9, (2b), (3b))

$$C(m) = (-1)^{m+1} M_{2,m} \qquad (m = 0, 1, \cdots) . \tag{6.6}$$

Thus, from (1.2), we obtain

$$M_{3,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+2} G_j^k \qquad (j = 0, 1, \cdots) \quad , \tag{6.7}$$

$$M_{2,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+1} G_j^k \qquad (j = 0, 1, \cdots) \quad . \tag{6.8}$$

Since $M_{3,j} = M_{2,j} + M_{1,j}$ for $j = 1, 2, \cdots$, we obtain from (6.7) and (6.8) that

$$M_{1,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k} G_j^k \qquad (j = 1, 2, \cdots) \quad . \tag{6.9}$$

Since $F_{2k+i-1} = F_{i-1}F_{2k+1} + F_{i-2}F_{2k}$ (see (1.10)), we obtain from [9, (6b)], using (6.8) and (6.9), that

$$M_{i,j} = (-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+i-1} G_j^k - \sum_{k=0}^{i-4} (k+1)^j F_{i-3-k} \qquad (j = 1, 2, \cdots) \quad . \tag{6.10}$$

From (6.4) and (6.10), we conclude that

$$(-1)^j \sum_{k=0}^{j} (-1)^k (k!) F_{2k+i-1} G_j^k = \sum_{k=0}^{j} k! F_{k+i} G_j^k \qquad (j = 1, 2, \cdots ; \; i = 0, 1, \cdots) . \tag{6.11}$$

It should be noted that [9, (7c)] was obtained from [9, (6a)], using [9, (7a)]. Since [9, (7c)] is a linear difference equation of second order in i, its solution is

$$(6.12)\quad P_i(m,n) = F_{i-1}P_2(m,n) + F_{i-2}P_1(m,n) - \sum_{k=0}^{i-3}(n-k)^m F_{i-1-k} \qquad (i = 3, 4, \cdots).$$

Using (6.12) and (1.10), [9, (8)] can be simplified to

$$(6.13)\quad S(m, n-h) = F_n P_1(m,n) + F_{n+1}P_2(m,n) + (-1)^{m+1}M_{2,m} - \sum_{k=0}^{h-2}(n-k)^m F_{n-k+1} - (n+1-h)^m F_{n+1-h} \qquad (h = 2, 3, \cdots).$$

Since $P_3^\star(m,n) = (-1)^m P_3(m,-n)$ [9, (9)] can be simplified (using [9, (6a), (7c)]) to

$$(6.14)\quad \sum_{k=1}^{n}(n-k+1)^m F_k = M_{1,m}F_{n+1} + M_{2,m}F_{n+2} + n^m + (-1)^{m+1}(P_2(m,-n) + P_1(m,-n)) \qquad (m = 1, 2, \cdots).$$

Since (see [9, (11)]) $P_i(m,n) = (-1)^m Q(m, -n+i-1)$, where $Q(m,n)$ are the Weinshenk polynomials in n of degree m (see reference [8] cited in [9]), it follows that

$$(6.15)\qquad Q(m,n) = (-1)^m P_1(m,n) = \sum_{j=0}^{m}\binom{m}{j} M_{1,j}n^{m-j}.$$

Thus (6.15), where $M_{1,k}$ is defined by (6.1), affords a closed form for the coefficients of $Q(m,n)$. From (6.12), with n replaced by $-n$, we obtain the following recursion relation for the Weinshenk polynomials:

$$(6.16) \qquad Q(m, n + i - 1) = F_{i-1}Q(m, n + 1) + F_{i-2}Q(m, n) - \sum_{k=0}^{i-3} (n + k)^m F_{i-1-k} \qquad (i = 3, 4, \cdots) .$$

In [9, (7a)] there is defined

$$(6.17) \qquad P_i(m, n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{i,j} n^{m-j} \qquad (m = 0, 1, \cdots) .$$

If we apply the well-known inverse pair relations,

$$(6.18) \qquad A_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_k , \qquad B_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_k$$

to (6.17), we obtain as its inverse

$$(6.19) \qquad M_{i,m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} P_i(j, n) n^{m-j} \qquad (m = 0, 1, \cdots) .$$

Since $P_i(j, n) = (-1)^j Q(j, -n + i - 1)$, we obtain from (6.19)

$$(6.20) \qquad M_{i,m} = \sum_{j=0}^{m} \binom{m}{j} Q(j, -n + i - 1) n^{m-j} .$$

From (1.19), we obtain for $n = 0$, recalling (6.9),

$$(6.21) \qquad (-1)^m (m!) F_{2m} = \sum_{j=1}^{m} (-1)^j S_m^j M_{1,j} \qquad (m = 1, 2, \cdots) .$$

From (1.20), we obtain for $n = 0$, recalling (6.8),

$$(-1)^m (m!) F_{2m+1} = \sum_{j=0}^{m} (-1)^j S_m^j M_{2,j} \quad (m = 0, 1, \cdots) . \tag{6.22}$$

From (2.35), we obtain, recalling (6.9),

$$m!\, F_{m+1} = \sum_{j=1}^{m} S_m^j M_{1,j} \quad (m = 1, 2, \cdots) . \tag{6.23}$$

From (2.36), we obtain, recalling (6.8),

$$m!\, F_{m+2} = \sum_{j=0}^{m} S_m^j M_{2,j} \quad (m = 0, 1, \cdots) . \tag{6.24}$$

If we set $b = 2$ in (4.3), then $U_n = (-1)^n$ is a solution of (4.3). In (4.20), set $P(k) = k^m$ so that $a_m = 1$, $a_j = 0$, $j = 0, 1, \cdots, m - 1$. Thus, (4.20), with $b = 2$ and $r = 0$, gives a closed form for

$$\sum_{k=0}^{n-1} (-1)^k k^m .$$

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