FIBONACCI AND LUCAS NUMBERS IN THE SEQUENCE OF GOLDEN NUMBERS

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Beginning with the golden rectangle with base 2 and altitude $\sqrt{5} - 1$, one may proceed to construct a sequence of numbers which represent altitudes (shortest sides) of the nested golden rectangles.

(1) $\sqrt{5} - 1, \; 3 - \sqrt{5}, \; 2\sqrt{5} - 4, \; 7 - 3\sqrt{5}, \; 5\sqrt{5} - 11, \; 18 - 8\sqrt{5}, \; \cdots$

We shall call this the sequence of golden numbers. These numbers, as one may suspect, are closely related to Fibonacci numbers, as is suggested by Theorem 2 below. First, however, we need to observe that the n^{th} golden number may be expressed by the following recursive formula:

Theorem 1. If g_n denotes the n^{th} golden number, then $g_n = 1/2 \, g_1 \cdot g_{n-1}$.

Proof. This follows immediately from the method of finding the altitude of a golden rectangle given its base (details left for the reader).

As an immediate consequence we have a corollary:

$$g_n = \frac{(\sqrt{5} - 1)^n}{2^{n-1}} \quad .$$

We next observe after considering the first few golden numbers that

Theorem 2. $\quad g_n = g_{n-2} - g_{n-1}$

Proof. Using the form for g_n given in the Corollary to Theorem 1, we have

$$g_{n-2} - g_{n-1} = \frac{(\sqrt{5} - 1)^{n-2}}{2^{n-3}} - \frac{(\sqrt{5} - 1)^{n-1}}{2^{n-2}}$$

$$= \frac{2^2 \cdot (\sqrt{5} - 1)^{n-2}}{2^{n-1}} - \frac{2 \cdot (\sqrt{5} - 1)^{n-1}}{2^{n-1}} = \frac{2(\sqrt{5} - 1)^{n-2}[2 - \sqrt{5} + 1]}{2^{n-1}}$$

$$= \frac{2(\sqrt{5} - 1)^{n-2} \cdot (3 - \sqrt{5})}{2^{n-1}} = \frac{2(\sqrt{5} - 1)^{n-2} \cdot \frac{(\sqrt{5} - 1)^2}{2}}{2^{n-1}} = \frac{(\sqrt{5} - 1)^n}{2^{n-1}}$$

$$= g_n \quad .$$

Another rather interesting observation is that the coefficients of radical 5 appear to be the sequence of Fibonacci numbers with alternating signs. We may formalize the conjecture after observing that as a result of a multiplication by $(\sqrt{5} - 1)/2$, the signs of each term of the golden numbers alternate and the n^{th} golden number may be expressed in the form

$$g_n = (-1)^{n-1} a_n \cdot \sqrt{5} - b_n$$

where a_n and b_n are positive integers.

Theorem 3. If

$$g_n = (-1)^{n-1}[a_n \cdot \sqrt{5} - b_n]$$

represents the n^{th} golden number, then a_n is the n^{th} Fibonacci number, F_n.

Proof.

$$g_{n+1} = g_n \cdot \frac{\sqrt{5} - 1}{2} = \frac{(-1)^{n-1}}{2} [5a_n - \sqrt{5}b_n - \sqrt{5}a_n + b_n]$$

$$= (-1)^n \left[\frac{(a_n + b_n)}{2} \sqrt{5} - \frac{(5a_n + b_n)}{2}\right]$$

$$\therefore a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \frac{5a_n + b_n}{2}$$

Then

$$a_{n-1} + a_n = a_{n-1} + \frac{a_{n-1} + b_{n-1}}{2} = \frac{3a_{n-1} + b_{n-1}}{2}$$

and

$$a_{n+1} = \frac{a_n + b_n}{2} = \frac{\dfrac{a_{n-1} + b_{n-1}}{2} + \dfrac{5a_{n-1} + b_{n-1}}{2}}{2} = \frac{3a_{n-1} + b_{n-1}}{2}$$

$$= a_{n-1} + a_n \longrightarrow a_n = F_n .$$

Yet another observation may be made from the sequence (1). It is stated:

Theorem 4. If g_n and g_{n+1} are any two successive golden numbers, then $F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = 2$.

Proof. Using the representation for F_{n+1} developed in the proof of Theorem 3, we write

$$F_{n+1} = \frac{F_n + b_n}{2} \rightarrow b_n = 2F_{n+1} - F_n$$

Therefore, we may express g_n and g_{n+1} in terms of Fibonacci numbers only:

$$g_n = (-1)^{n-1}(F_n\sqrt{5} + F_n - 2F_{n+1})$$

and

$$g_{n+1} = (-1)^n(F_{n+1}\sqrt{5} + F_{n+1} - 2F_{n+2}) .$$

Thus we obtain:

$$F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = (-1)^{n-1}[F_n \cdot F_{n+1}\sqrt{5} + F_n \cdot F_{n+1} - 2F^2_{n+1}$$

$$- F_n \cdot F_{n+1}\sqrt{5} - F_n \cdot F_{n+1} + 2F_n \cdot F_{n+2}]$$

Recalling the fundamental identity

$$F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n, \qquad n \geq 2,$$

it follows that

$$F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = (-1)^{n-1}[-2F_{n+1}^2 + 2(F_{n+1}^2 + (-1)^{n+1})] = 2$$

Recalling the representation for g_n used in the proof of Theorem 4,

$$g_n = (-1)^{n-1}(F_n\sqrt{5} + F_n - 2F_{n+1})$$

we observe that

$$F_n - 2F_{n+1} = F_n - 2[F_{n-1} + F_n] = -F_n - 2F_{n-1}$$

which gives us the following alternate forms for the n^{th} golden number:

$$g_n = (-1)^{n-1}(F_n \cdot g_1 - 2F_{n-1})$$

or

$$g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n)$$

where L_n is the n^{th} Lucas number. We now state our final result.

Theorem 5. $g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n)$

Proof. Follows from the identity

$$L_n = F_{n-1} + F_{n+1} \quad .$$

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