EQUATIONS WHOSE ROOTS ARE THE nth POWERS OF THE ROOTS OF A GIVEN CUBIC EQUATION

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Given the cubic equation

$$x^3 - c_1x^2 + c_2x - c_3 = 0 \quad ,$$

with roots r_1, r_2, r_3, the problem of this paper is to write the equation

$$(1) \quad x^3 - (r_1^n + r_2^n + r_3^n)x^2 + (r_1^n r_2^n + r_1^n r_3^n + r_2^n r_3^n)x - r_1^n r_2^n r_3^n$$
$$= x^3 - c_{(1,n)}x^2 + c_{(2,n)}x - c_{(3,n)} = 0$$

whose roots are r_1^n, r_2^n, r_3^n, and whose coefficients are expressed in terms of the coefficients c_1, c_2, c_3, of the given equation.

This paper extends to the cubic equation a study initiated by the solution of a similar problem for the quadratic by the same authors [1]. Just as a special quadratic equation leads to a relationship between the n^{th} Fibonacci number and a sum of binomial coefficients, so does a special cubic equation relate the n^{th} member of a Tribonacci sequence to a sum of products of binomial coefficients. Some Lucas identities also follow.

The summation for initial values of powers of roots by elementary theory yields the first five entries in the table below. Examination of this sequence reveals an iterative pattern; namely, that if

$$c_{(1,n)} = r_1^n + r_2^n + r_3^n, \qquad n \geq 0 \; ,$$

then

$$c_{(1,n)} = c_1c_{(1,n-1)} - c_2c_{(1,n-2)} + c_3c_{(1,n-3)} \quad ,$$

which is easily proved, since each root, and hence sums of the roots, satisfies the original equation.

Sums through eighth powers appear in the table below. The right-hand column gives the sum of the absolute values of the coefficients for each value of n.

n	$c_{(1,n)} = r_1^n + r_2^n + r_3^n$	Coefficient Sums for n
0	3	3
1	c_1	1
2	$c_1^2 - 2c_2$	3
3	$c_1^3 - 3c_1c_2 + 3c_3$	7
4	$c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3$	11
5	$c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3$	21
6	$c_1^6 - 6c_1^4c_2 + 9c_1^2c_2^2 - 2c_2^3 + 6c_1^3c_3 - 12c_1c_2c_3 + 3c_3^2$	39
7	$c_1^7 - 7c_1^5c_2 + 14c_1^3c_2^2 - 7c_1c_2^3 + 7c_1^4c_3 - 21c_1^2c_2c_3 + 7c_2^2c_3 + 7c_1c_3^2$	71
8	$c_1^8 - 8c_1^6c_2 + 20c_1^4c_2^2 - 16c_1^2c_2^3 + 2c_2^4$	131
	$+ 8c_1^5c_3 - 32c_1^3c_2c_3 + 24c_1c_2^2c_3 + 12c_1^2c_3^2 - 8c_2c_3^2$	

It is possible to perceive the generalized number pattern for sums of n^{th} powers of the roots by extending the table above and breaking down the sum in terms of coefficients of powers of c_3. If ψ_n is the coefficient of $c_3^n/n!$ in the sum

$$r_1^n + r_2^n + r_3^n \quad ,$$

then

$$\psi_0 = c_1^n - nc_1^{n-2}c_2 + n(n-3)c_1^{n-4}c_2^2/2! - n(n-4)(n-5)c_1^{n-6}c_2^3/3! + \cdots$$

$$\psi_1 = nc_1^{n-3} - n(n-4)c_1^{n-5}c_2 + n(n-5)(n-6)c_1^{n-7}c_2^2/2!$$
$$- n(n-6)(n-7)(n-8)c_1^{n-9}c_2^3/3! + \cdots$$

$$\psi_2 = n(n-5)c_1^{n-6} - n(n-6)(n-7)c_1^{n-8}c_2 + n(n-7)(n-8)(n-9)c_1^{n-10}c_2^2/2!$$
$$- n(n-8)(n-9)(n-10)(n-11)c_1^{n-12}c_2^3/3! + \cdots$$

leading to the three equivalent expressions below. For $c_1c_2c_3 \neq 0$,

$$(2) \qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} c_3^k \psi_k / k! \quad ,$$

$$\psi_k = \sum_{m=0}^{[(n-3k)/2]} (-1)^m c_1^{n-2m-3k} c_2^m n(n - m - 2k - 1)!/(n - 2m - 3k)!\, m! \ ,$$

$$(3) \qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n(n - m - 2k - 1)!}{(n - 2m - 3k)!\, m!\, k!} c_1^{n-2m-3k} c_2^m c_3^k \ ,$$

$$(4) \qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n - m - 3k} \binom{n-m-2k-1}{k} \binom{n-m-3k}{m} \times c_1^{n-2m-3k} c_2^m c_3^k \ ,$$

where $[n]$ is the greatest integer $\leq n$ and $\binom{m}{n}$ is a binomial coefficient. Notice that the coefficients of c_3^0 are the same as the coefficients which arose in studying the roots of the quadratic in [1]. The reiterative relationship of the terms $c_{(1,n)}$ suggests a proof by mathematical induction for the three formulae listed, and such a proof has been written by the authors. For the sake of brevity, the proof is omitted. A derivation of the above formulas could also be done using Waring's formula and Newton's identities (see [2]).

Thus far, we have found a way to express the coefficient for x^2 in our general problem. The coefficient for x,

$$c_{(2,n)} = r_1^n r_2^n + r_1^n r_3^n + r_2^n r_3^n \ ,$$

has a similar computation. In the auxiliary cubic equation

$$(x - r_1^n r_2^n)(x - r_1^n r_3^n)(x - r_2^n r_3^n) = 0$$

notice that $c_{(2,n)}$ is the coefficient of x^2. When $n = 1$, the above cubic becomes, upon multiplication,

$$x^3 - (r_1r_2 + r_1r_3 + r_2r_3)x^2 + (r_1^2r_2r_3 + r_1r_2^2r_3 + r_1r_2r_3^2)x - r_1^2r_2^2r_3^2$$
$$= x^3 - c_2x^2 + c_1c_3x - c_3^2 = 0.$$

Comparing this equation with the equations of our original problem, we can apply the three formulae already derived for $c_{(1,n)}$ to find $c_{(2,n)}$ if we replace c_1 by c_2, c_2 by c_1c_3, and c_3 by c_3^2. For example, from Equation (4), if $c_1c_2c_3 \neq 0$, our formula for $c_{(2,n)}$ becomes

$$(5) \qquad r_1^nr_2^n + r_1^nr_3^n + r_2^nr_3^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n(n-m-2k-1)!}{(n-2m-3k)!\,m!\,k!} \times c_1^m c_2^{n-2m-3k} c_3^{2k+m}$$

In practice, when raising the roots of a given equation, it is simpler to utilize the method of iterating functions than to substitute into the formulae, especially as n becomes larger. An example worked by each method follows.

Given the equation

$$x^3 - 6x^2 + 11x - 6 = 0 \ ,$$

write the cubic whose roots are the fourth powers of the roots of the given equation, without solving for the roots.

(A) By substitution: From the table given earlier or from Equations (3) and (5), the desired cubic is

$$x^3 - (c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3)x^2 + (c_2^4 - 4c_2^2c_1c_3 + 2c_1^2c_3^2 + 4c_2c_3^2)x - c_3^4 = 0.$$

Substituting

$$c_1 = 6, \quad c_2 = 11, \quad c_3 = 6$$

yields

$$x^3 - 98x^2 + 1393x - 6^4 = 0$$

with roots 1^4, 2^4, and 3^4. As a check, the roots of the given equation are 1, 2, and 3.

(B) By iteration: To get $c_{(1,4)}$, we wish to write the sequence

$$c_{(1,0)},\ c_{(1,1)},\ c_{(1,2)},\ c_{(1,3)},\ c_{(1,4)}.$$

Now

$$c_{(1,0)} = 3,\quad c_{(1,1)} = c_1 = 6,\quad c_{(1,2)} = c_1^2 - 2c_2 = 36 - 22 = 14\,.$$

By the iteration relationship,

$$c_{(1,3)} = c_1c_{(1,2)} - c_2c_{(1,1)} + c_3c_{(1,0)} = 6(3) - 11(6) + 6(14) = 36\ ;$$

$$c_{(1,4)} = 6(6) - 11(14) + 6(36) = 98\ .$$

Similarly,

$$c_{(2,0)} = 3,\quad c_{(2,1)} = 11,\quad c_{(2,2)} = c_2^2 - 2c_1c_3 = 49.$$

Since

$$c_{(n,2)} = c_2c_{(2,n-1)} - c_1c_3c_{(2,n-2)} + c_3^2c_{(2,n-3)}$$

substitution yields

$$c_{(2,3)} = 251 \text{ and } c_{(2,4)} = 1393\ ,$$

yielding the same cubic as in (A).

Next let us turn our attention to several special cubic equations. First, for the cubic

$$x^3 - x^2 - x - 1 = 0,\qquad c_{(1,0)} = 3,\quad c_{(1,1)} = 1,\quad c_{(1,2)} = 3,$$

and our repeating multipliers in the iterative relationship are 1, 1, 1. Then,

$$c_{(1,3)} = 1(3) + 1(1) + 1(3) = 7, \quad c_{(1,4)} = 7 + 3 + 1 = 11,$$

$$\ldots, \quad c_{(1,n)} = c_{(1,n-1)} + c_{(1,n-2)} + c_{(1,n-3)} .$$

For this particular equation we have a species of Tribonacci numbers, any term after the third being the sum of the three preceding terms, with the entry terms 3, 1, 3. By Equation (4), the n^{th} term T_n in this Tribonacci sequence is

$$T_n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n - m - 3k} \binom{n - m - 2k - 1}{k} \binom{n - m - 3k}{m}$$

Notice that the sums of the coefficients in the table given for $c_{(1,n)}$ are these same numbers. It is interesting to recall that the special equation

$$x^2 - x - 1 = 0$$

led to a formula relating the n^{th} member of the Fibonacci sequence to a sum of binomial coefficients in the earlier study of the quadratic equation [1].

Considering the special equation

$$x^3 - x^2 + x - 1 = 0$$

with roots $1, \pm\sqrt{-1}$, we can write the following from Equation (4):

$$\sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^m n}{n - m - 3k} \binom{n - m - 2k - 1}{k} \binom{n - m - 3k}{m} = \begin{cases} 1 \text{ if } n \text{ is odd} \\ -1 \text{ if } n = 4s+2 \\ 3 \text{ if } n = 4s \end{cases}$$

Of more interest, however, are the following identities for the n^{th} Lucas number L_n, defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2} \quad .$$

We substitute in Equation (3), using

$$r_1 = \alpha = (1 + \sqrt{5})/2, \quad r_2 = \beta = (1 - \sqrt{5})/2 \quad ,$$

and letting r_3 vary. If $r_3 = 1$, Equation (3) cannot be used directly because

$$c_2 = \alpha\beta + \beta + \alpha = 0,$$

and 0^0 is not defined, But, by following the derivation for Equation (3), it is seen that, if $c_2 = 0$, $c_1c_3 \neq 0$,

$$(3') \qquad r_1^n + r_2^n + r_3^n = \sum_{k=0}^{[n/3]} \frac{n(n - 2k - 1)!}{(n - 3k)!\,k!} c_1^{n-3k} c_3^k$$

Since

$$c_1 = \alpha + \beta + 1 = 2, \quad c_3 = \alpha\beta = -1,$$

and

$$L_n = \alpha^n + \beta^n \quad ,$$

substitution gives

$$L_n + 1 = \sum_{k=0}^{[n/3]} \frac{(-1)^k 2^{n-3k} n(n - 2k - 1)!}{(n - 3k)!\; k!}$$

In general, if

$$r_3 = p, \quad p \neq 1, \quad p \neq -1, \quad p \neq 0 \, ,$$

Equation (3) gives

$$L_n + p^n = \sum_{k=0}^{[n/3]} \sum_{m=0}^{[(n-3k)/2]} \frac{(-1)^{m+k} n(n-m-2k-1)!\,(p+1)^{n-2m-3k}(p-1)^m p^k}{(n-2m-3k)!\; m!\; k!}$$

Similarly, Equation (3') gives the following two identities using

$$r_1 = \alpha, \quad r_2 = \beta, \quad r_3 = -1/\sqrt{5}\ ,$$

and the known relationship for Fibonacci numbers,

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}\quad .$$

Below, n is taken to be $2s+1$ and $2s$ respectively.

$$F_{2s+1} - 1/5^{s+1} = \sum_{k=0}^{[(2s+1)/3]} \frac{(2s+1)\,(2s-2k)!\,(-1)^k 4^{2s-3k+1}}{(2s-3k+1)!\,k!\,5^{s-k+1}}$$

$$L_{2s} + 1/5^s = \sum_{k=0}^{[2s/3]} \frac{2s(2s-2k-1)!\,(-1)^k\,4^{2s-3k}}{(2s-3k)!\;k!\;5^{s-k}}$$

REFERENCES

1. N. A. Draim and M. Bicknell, "Sums of the n^{th} Powers of the Roots of a Given Quadratic Equation," The Fibonacci Quarterly, 4:2, April, 1966, pp. 170-178.
2. W. S. Burnside and A. W. Panton, An Introduction to the Theory of Binary Algebraic Forms, Dover, New York, 1960, Chapter 15.

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