RESTRICTED COMPOSITIONS

S. G. MOHANTY
McMaster University, Hamilton, Ontario, Canada

As a continuation of [6] and [7], this paper deals with a restricted set of compositions of an integer (to be defined below) and presents extensions of some results of Gould [2], [3], [4], by interpreting the compositions through the corresponding lattice paths.

By the definition in [7], a $(k+1)$-composition $(t_1, t_2, \cdots, t_{k+1})$ of an integer n (i. e.,

$$\sum_{i=1}^{k+1} t_i = n \quad \text{and} \quad t_i \geq 1$$

for every i) dominates another $(k+1)$-composition $(t_1', t_2', \cdots, t_{k+1}')$ of n if and only if

$$\sum_{i=1}^{j} t_i \geq \sum_{i=1}^{j} t_i' \quad \text{for} \quad j = 1, 2, \cdots, k+1 \quad .$$

Using the 1:1 correspondence in [6], we associate with each $(k+1)$-composition of n a minimal lattice path (onward and upward path through lattice points) from $(0, 0)$ to $(n-k-1, k)$ such that the directed distance measured along the positive direction of x-axis, of the point $(n-k-1, k-j)$, $j = 1, 2, \cdots, k$ from the path is

$$\sum_{i=1}^{j} t_i - j \quad .$$

Without any ambiguity, denote this path by

$$\left[t_1 - 1,\ t_1 + t_2 - 2, \cdots, \sum_{i=1}^{k} t_i - k\right].$$

Thus, it is evident that to the set $C(n; a_1, a_2, \cdots, a_k)$ of $(k+1)$-compositions of n, dominated by the $(k+1$-composition $(a_1, a_2, \cdots, a_{k+1})$ of n corresponds the set $L(A_1, A_2, \cdots, A_k)$ of lattice paths which do not cross to the left or above the path

$$\left[A_1, A_2, \cdots, A_k\right] = \left[a_1 - 1,\ a_1 + a_2 - 2, \cdots, \sum_{i=1}^{k} a_i - k\right] .$$

Let the number in the set C (equivalently in L) be represented by $N(n; a_1, a_2, \cdots, a_k)$ for $k \geq 1$, and by $N(n)$ for $k = 0$. Trivially,

$$N(n) = 1 \quad , \tag{1}$$

$$N(n; a, \underbrace{1, 1, \cdots, 1}_{k-1}) = \binom{a + k - 1}{k} \quad , \tag{2}$$

and

$$N(n; a_1, a_2, \cdots, a_k) = 0 \quad , \tag{3}$$

if any a_i is either zero or negative.

Now consider the path

$$\left[A_1', A_2', \cdots, A_k'\right]$$

such that $A_i' \leq A_i$ for all i. Every path in L passes through one of the points $(n - k - A_{i+1}' - 2,\ k - i)$, $i = 0, 1, 2, \cdots, k$, $(A_{k+1}' = A_k')$ before moving to $(n - k - A_{i+1}' - 1,\ k - i)$ and then reaches $(n - k - 1, k)$ not crossing $[A_1', A_2', \cdots, A_k']$. Therefore,

$$\begin{aligned}(4)\qquad N(n;a_1, a_2,\cdots, a_k) = {} & N(n;a_1 - a_1', a_2,\cdots, a_k)N(n)\\ & + N(n;a_1 + a_2 - a_1' - a_2', a_3,\cdots, a_k)N(n;a_1')\\ & + N(n;a_1 + a_2 + a_3 - a_1' - a_2' - a_3', a_4,\cdots, a_k)N(n;a_1', a_2')\\ & + \cdots + N\left(n;\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} a_i'\right)N(n;a_1', a_2',\cdots, a_{k-1}')\\ & + N(n)N(n;a_1', a_2',\cdots, a_k') \quad .\end{aligned}$$

We note that whenever $A_i' = A_i$,

$$N(n;a_1 + \cdots + a_i - a_1' - \cdots - a_i', a_{i+1},\cdots, a_k) = 0 \quad .$$

It may be pointed out that relation (4) in some sense is a generalization of Vandermonde's convolution

$$\sum_{i=0}^{k} \binom{x}{i}\binom{y}{k-i} = \binom{x+y}{k} ,$$

a further discussion of which is given later.

By setting $a_1 = A_k' + 1$ and $a_2 = a_3 = \cdots = a_k = 1$ in (4) and using (2), we get the recursive formula

$$(5)\qquad N(n;a_1', a_2',\cdots, a_k') =$$

$$\binom{A_k' + k}{k} - \sum_{i=1}^{k-1} \binom{A_k' - A_i' + k - i}{k - i + 1} N(n;a_1', a_2',\cdots, a_{i-1}')$$

which is the same as (9) in [1] and (2) in [8]. The solution of (5) is stated in the following theorem.

Theorem 1:

(6) $N(n; a_1, a_2, \cdots, a_k)$

$$= \begin{vmatrix} \binom{A_k + k}{k} & \binom{A_k - A_{k-1} + 1}{2} & \binom{A_k - A_{k-2} + 2}{3} & \cdots & \binom{A_k - A_1 + k - 1}{k} \\ \binom{A_{k-2} + k - 2}{k - 2} & 1 & \binom{A_{k-2} - A_{k-2}}{1} & \cdots & \binom{A_{k-2} - A_1 + k - 3}{k - 2} \\ \binom{A_{k-3} + k - 3}{k - 3} & 0 & 1 & \cdots & \binom{A_{k-3} - A_1 + k - 4}{k - 3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \binom{A_1 + 1}{1} & 0 & 0 & \cdots & \binom{A_1 - A_1}{1} \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

Another way of expressing the number in L leads to

$$(7) \qquad N(n; a_1, a_2, \cdots, a_k) = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1$$

$$= \sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1$$

$$+ \sum_{x_1=\alpha}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=x_{k-1}}^{A_k} 1, 0 \le \alpha \le A_1 + 1 .$$

Substituting $x_i - \alpha = x_i'$ for $i = 1, 2, \cdots, k$, the second term on the right hand side becomes

$$(8) \qquad \sum_{x_1'=0}^{A_1-\alpha} \sum_{x_2'=x_1'}^{A_2-\alpha} \cdots \sum_{x_k'=x_{k-1}'}^{A_k-\alpha} 1 = N(n; a_1 - \alpha, a_2, \cdots, a_k) .$$

On the other hand, the first term can be written as

$$(9)\qquad \sum_{x_1=0}^{\alpha-1}\sum_{x_2=x_1}^{A_2}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 = \sum_{x_1=0}^{\alpha-1}\sum_{x_2=0}^{A_2}\sum_{x_3=x_1}^{A_3}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1$$

$$- \sum_{x_1=1}^{\alpha-1}\sum_{x_2=0}^{x_1-1}\sum_{x_3=x_1}^{A_3}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 ,$$

whereas the last term in (9) can again be expressed as

$$-\sum_{x_1=1}^{\alpha-1}\sum_{x_2=0}^{x_1-1}\sum_{x_3=0}^{A_3}\sum_{x_4=x_3}^{A_4}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 + \sum_{x_1=2}^{\alpha-1}\sum_{x_2=1}^{x_1-1}\sum_{x_1=0}^{x_2-1}\sum_{x_4=x_3}^{A_4}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1.$$

When we proceed in the above manner, the final expression for (9) is

$$(10)\qquad \sum_{i=1}^{k}(-1)^{i+1}\binom{\alpha}{i} N(n;A_{i+1}+1,\ a_{i+2},\cdots,a_k)\ ,$$

by noting that

$$\sum_{x_1=i-1}^{\alpha-1}\sum_{x_2=i-2}^{x_1-1}\cdots\sum_{x_i=0}^{x_{i-1}-1} 1 = \sum_{x_1=0}^{\alpha-i}\sum_{x_2=0}^{x_1}\cdots\sum_{x_i=0}^{x_{i-1}} 1 = \binom{\alpha}{i}$$

and

$$\sum_{x_{i+1}=0}^{A_{i+1}}\sum_{x_{i+2}=x_{i+1}}^{A_{i+2}}\cdots\sum_{x_k=x_{k-1}}^{A_k} 1 = N(n;A_{i+1}+1,\ a_{i+2},\cdots,a_k).$$

Thus it follows from (7), (8) and (10) that

$$(11)\quad \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} N(n; A_{i+1} + 1, a_{i+2}, \cdots, a_k) = N(n; a_1 - \alpha, a_2, \cdots, a_k) .$$

An alternative way of simplifying the first term on the right of (9) is

$$\sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=0}^{A_k} 1 - \sum_{x_1=0}^{\alpha-1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_k=0}^{x_{k-1}-1} 1 ,$$

where the sums in the last term for which $x_{k-1} - 1$ is negative are zero. Repetition of this process yields

$$(12)\quad \sum_{i=1}^{k} (-1)^{i+1} \binom{A_{k+1-i} + 1}{i} N(n; a_1, a_2, \cdots, a_{k-i}) = N(n; a_1, a_2, \cdots, a_k)$$

for $c = a_1$. Relation (12) has been obtained earlier in [7], which is equivalent to (3) in [1].

When $c = a_1$, the solution of either (11) or (12) is stated as Theorem 2, for which a direct elementary proof is provided below.

Theorem 2:

$$(13)\quad N(n; a_1, a_2, \cdots, a_k) = \begin{vmatrix} \binom{A_k + 1}{1} & \binom{A_{k-1} + 1}{2} & \cdots & \binom{A_1 + 1}{k} \\ 1 & \binom{A_{k-1} + 1}{1} & \cdots & \binom{A_1 + 1}{k - 1} \\ 0 & 1 & \cdots & \binom{A_1 + 1}{k - 2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

Proof: Obviously

$$\begin{vmatrix} \binom{x_k}{0} & \binom{x_{k-1}}{1} & \binom{x_{k-2}}{2} & \cdots & \binom{x_1}{k} \\ 0 & 1 & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix} = 1 \quad .$$

Using this in (7), we see that

$$N(n;a_1, a_2, \cdots, a_k) = \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_{k-1}=x_{k-2}}^{A_{k-1}} \begin{vmatrix} \binom{A_k+1}{1} - \binom{x_{k-1}}{1} & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k} \\ \binom{A_k+1}{0} - \binom{x_{k-1}}{0} & 1 & \cdots & \binom{x_1}{k-1} \\ 0 & 0 & \cdots & \binom{x_1}{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix}$$

$$= \sum_{x_1=0}^{A_1} \sum_{x_2=x_1}^{A_2} \cdots \sum_{x_{k-1}=x_{k-2}}^{A_{k-1}} \begin{vmatrix} \binom{A_k+1}{1} & \binom{x_{k-1}}{1} & \cdots & \binom{x_1}{k} \\ 1 & 1 & \cdots & \binom{x_1}{k-1} \\ 0 & 0 & \cdots & \binom{x_1}{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{x_1}{0} \end{vmatrix}$$

The proof is complete when the summation is continued to the end.

Theorems 1 and 2 give rise to an interesting combinational identity on determinants, the direct proof of which is not obvious.

We check either from the theorems or directly that

$$\text{(14)} \quad N(n;a_1, a_2, \cdots, a_k) + N(n;a_1 + a_2, a_3, \cdots, a_k) = N(n;a_1 + 1, a_2, \cdots, a_k) \ ,$$

$$\text{(15)} \quad N(n;1, a_2, \cdots, a_k) = N(n;a_2, a_3, \cdots, a_k) \ ,$$

and

$$\text{(16)} \quad N\left(\sum_{i=1}^{k} a_i + j;\ a_1, a_2, \cdots, a_k\right) = N\left(\sum_{i=1}^{k} a_i + 1;\ a_1, a_2, \cdots, a_k\right) \quad j = 1, 2, \cdots$$

A few important special cases are considered below.

Corollary 1.

$$\text{(17)} \quad N(n;a, \underbrace{b, \cdots, b}_{k-1}) = \frac{a}{a + kb}\binom{a + kb}{k} \ .$$

This is directly verifiable from either one of the theorems. (Also see Theorem 1 in [6]).

In the next, we evaluate

$$N_{p,q}(a, b; c, d) = N(n;a, \underbrace{b, \cdots, b}_{p-1}, c, \underbrace{d, \cdots, d}_{q-1})$$

which has been obtained by a different method as Theorem 3 in [6].

Corollary 2.

$$\text{(18)} \quad N_{p,q}(a, b;c, d) = \sum_{i=0}^{q} (-1)^i \frac{a}{a + (p + q - i)b}\binom{a + (p + q - i)b}{p + q - i}$$

$$\times \frac{(q - i + 1)b - c - (q - i)d}{(q - i + 1)b - c - qd + i}\binom{(q - i + 1)b - c - qd + i}{i} \ .$$

Proof: For $c + q(d-1) \geq qb$, the result is immediate, by taking $A_i' = (a-1) + (i-1)(b-1)$, $i = 1, 2, \cdots, p+q$ in (4) and applying Corollary 1. When $c + q(d-1) < qb$, let $s(p < s \leq p+q)$ be the largest integer so that $c + s(d-1) \geq sb$. $N_{p,q}(a,b;c,d)$ and $N(n;a,\underbrace{b,\cdots,b}_{p+q-1})$, expressed with the help of (4), where

$$A_i' = \begin{cases} (a-1) + (i-1)(b-1) & i = 1, 2, \cdots, s, \\ (a-1) + (p-1)(b-1) + (c-1) + (i-p-1)(d-1) & i = s+1, \\ & \quad s+2, \cdots, p+q, \end{cases}$$

lead to (18), after some simplification.

For completeness, we present two more special cases which are known and can easily be derived.

Corollary 3:

$$(19) \quad N_{p,q}(a,b;c,1) = \binom{a+c-2+(p-1)(b-1)+p+q}{p+q} - \sum_{i=q+1}^{p+q} \frac{a}{a+(p+q-i)b}\binom{a+(p+q-i)b}{p+q-i}\binom{c+q-(q-i+1)b-1}{i}.$$

Corollary 4:

$$(20) \qquad N_{p,q}(a,2;2,1) = \binom{a+2p+q-1}{p+q} - \binom{a+2p+q-1}{p-1}$$

In his paper [2], Gould has defined

$$A_k(\beta,\gamma) = \frac{\beta}{\beta+\gamma k}\binom{\beta+\gamma k}{k}$$

and has shown that $A_k(\beta,\gamma)$ satisfies the relation

$$(21) \qquad \sum_{i=0}^{k} A_i(\beta,\gamma)A_{k-i}(\delta,\gamma) = A_k(\beta+\delta,\gamma) .$$

Suppose that β, γ and δ are non-negative integers. Then (21) immediately follows from (4) and (17) by putting $a_1 = \beta + \delta$, $a_1 = a_3 = \cdots = a_k = \gamma$, $a_1' = \beta$, and $a_2' = a_3' = \cdots = a_k' = \gamma$ in (4). Relation (11) in [2] can similarly be verified. Also, the convolution (5.5) in [3] for $t = 0$ can be compared with (11) and their equivalence is easily established.

In what follows, the results on restricted compositions are analogous to those on unrestricted compositions in Gould's paper [4] (Theorems 1 and 5 or equivalently Theorem 6). Fix $a_2, a_3, \cdots, a_k$ and let

$$m = \sum_{i=2}^{k} a_i .$$

From (14), (15) and (16) we infer that

$$(22)\quad N(m + a_1 + 1;\ a_1, a_2, \cdots, a_k) = \sum_{i=1}^{a_1} N(m + i;\ a_2 - 1 + i, a_3, \cdots, a_k)$$

$$= \sum_{i=1}^{a_1} \left[\frac{m + a_1}{m + i}\right] N\{m + i;\ a_2 - 1 + i, a_3, \cdots, a_k\}$$

where $[z]$ is the greatest integer less than or equal to z and $N\{m + i; a_2 - 1 + i,\ a_3, \cdots, a_k\}$ is the number of compositions in the set $S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ which is defined as follows: For i negative or equal to zero,

$$S(m + i;\ a_2 - 1 + i, a_3, \cdots, a_k) \text{ is empty;}$$
$$S(m + 1;\ a_2, a_3, \cdots, a_k) = C(m + 1;\ a_2, a_3, \cdots, a_k) \ ;$$

For $i \geq 2$, $S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ is the subset of $C(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$ with the property that if $(x_1, x_2, \cdots, x_k) \in S(m + u; a_2 - 1 + u, a_3, \cdots, a_k)$, $u = 1, 2, \cdots, i - 1$, then for r a positive integer $(rx_1, rx_2, \cdots, rx_k) \notin S(m + i; a_2 - 1 + i, a_3, \cdots, a_k)$. Expression (22) corresponds to Theorem 1 in [4].

$$\sum_{j=1}^{\infty} N(m+j+1;\ j, a_2, \cdots, a_k)\, x^{m+j} = \sum_{j=1}^{\infty} x^{m+j}$$

$$\times \sum_{i=1}^{m} \left[\frac{m+j}{m+i}\right] N\{m+i;\ a_2 - 1 + i,\ a_3, \cdots, a_k\}$$

$$= \sum_{i=1}^{\infty} N\{m+i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \sum_{j=i}^{\infty} \left[\frac{m+j}{m+i}\right] x^{m+j}$$

$$= \sum_{i=1}^{\infty} N\{m+i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \frac{x^{m+i}}{(1-x)(1-x^{m+i})}$$

by (3) in [4]. Therefore,

$$(23)\quad \sum_{i=1}^{\infty} N\{m+i;\ a_2 - 1 + i, a_3, \cdots, a_k\} \frac{x^{m+i}}{(1-x^{m+i})}$$

$$= \sum_{i=1}^{\infty} N(m+i+1; i, a_2, \cdots, a_k) x^{m+i}(1-x)$$

$$= \sum_{i=1}^{\infty} N(m+i;\ a_2 - 1 + i,\ a_3, \cdots, a_k) x^{m+i}$$

by (14), (15) and (16). But (23) can be written as

$$(24)\quad \sum_{i=m+1}^{\infty} N\{i;\ a_2 - m - 1 + i, a_3, \cdots, a_k\} \frac{x^i}{1-x^i}$$

$$= \sum_{i=m+1}^{\infty} N(i; a_2 - m - 1 + i, a_3, \cdots, a_k) x^i \quad .$$

In order to extend the summation to $i = 1, 2, \cdots, m$ in (24), define

$$N^*(i; a_2 - m - 1 + i, a_3, \cdots, a_k) = \begin{cases} 0 \ \text{ for } \ i = 1, 2, \cdots, m \\ N(i; a_2 - m - 1 + i, a_3, \cdots, a_k) \text{ for } i = m+1, m+2 \cdots \end{cases}$$

Thus, following the procedure in [4],

$$(25)\quad N\{n; n + a_2 - m - 1, a_3, \cdots, a_k\} = \sum_{i|n} N^*(n; n + a_2 - m - 1, a_3, \cdots, a_k)\, \mu\left(\frac{n}{i}\right) ,$$

which is similar to that of Theorem 5 in [4].

We finally remark that such results can also be obtained for the number of lattice paths in the set $L_i(A_1, A_2, \cdots, A_k)$ defined as follows:

$L_0(A_1, A_2, \cdots, A_k) = L(A_1, A_2, \cdots, A_k)$; $L_i(A_1, A_2, \cdots, A_k)$ is the subset of $L(A_1 + i,\ A_2 + i, \cdots, A_k + i)$ such that if $[x_1, x_2, \cdots, x_k] \in L_u(A_1, A_2, \cdots, A_k)$, $u = 0, 1, \cdots, i - 1$, then $[rx_1, rx_2, \cdots, rx_k] \notin L_i(A_1, A_2, \cdots, A_k)$.

REFERENCES

1. F. Gobel, "Some Remarks on Ballot Problems," Report S321a, Math Centrum, Amsterdam, 1964.
2. H. W. Gould, "Some Generalizations of Vandermonde's Convolutions," Amer. Math. Monthly, Vol. 63, 1956, pp. 84-91.
3. H. W. Gould, "A New Convolution Formula and Some New Orthogonal Relations for Inversion of Series," Duke Math. Journal, Vol. 29, 1962,
4. H. W. Gould, "Binomial Coefficients, The Bracket Function, and Compositions with Relative Prime Summands," Fibonacci Quarterly, Vol. 2, 1964, pp. 241-260.
5. S. G. Mohanty, "Some Properties of Compositions and Their Application to the Ballot Problem," Can. Math. Bull. Vol. 8, 1965, pp. 359-372.
6. S. G. Mohanty and T. V. Narayana, "Some Properties of Compositions and Their Application to Probability and Statistics I," Biometrische Zeitschrift, Vol. 3, 1961, pp. 252-258.
7. T. V. Narayana and G. E. Fulton, "A Note on the Compositions of an Integer," Can. Math. Bull., Vol. 1, 1958, pp. 169-173.
8. P. Switzer, "Significance Probability Bounds for Rank Orderings," Ann. Math. Statist., Vol. 35, 1964, pp. 891-894.

* * * * *