ENUMERATION OF CERTAIN TRIANGULAR ARRAYS

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1. INTRODUCTION

Let k be a positive integer. We define the numbers $F_n(k)$ and $N_n(k)$ by means of the recurrences

$$F_n(k) = F_{n-1}(k) + F_{n-k}(k) \quad (n \geq k) , \tag{1.1}$$

$$N_{n-k}(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} N_{n-j}(k) \quad (n \geq k) , \tag{1.2}$$

with the initial conditions

$$F_n(k) = n + 1 \quad (0 \leq n < k) , \tag{1.3}$$

$$N_n(k) = \binom{k+n}{n} \quad (0 \leq n < k) , \tag{1.4}$$

Note that

$$F_n(1) = N_n(1) = 2^n , \tag{1.5}$$

$$F_n(2) = F_{n+2} , \tag{1.6}$$

$$N_n(2) = 3.2^{n-1} , \tag{1.7}$$

where F_j denotes the usual Fibonacci number $(F_0 = 0, \ F_1 = 1)$.

Given positive integers m and k, put $m = pk + r$ $(1 \leq r \leq k)$ and let $T(k, m)$ denote the number of arrays

$$\begin{array}{ccccccccc}
n_{11} & \cdots & n_{1k} & n_{1,k+1} & \cdots & n_{1,pk} & n_{1,pk+1} & \cdots & n_{1m} \\
 & & & n_{2,k+1} & \cdots & n_{2,pk} & n_{2,pk+1} & \cdots & n_{2m} \\
 & & & & & \vdots & \vdots & & \vdots \\
 & & & & & n_{p,pk} & n_{p,pk+1} & \cdots & n_{pm} \\
 & & & & & & n_{p+1,pk+1} & \cdots & n_{p+1,m}
\end{array} , \tag{1.8}$$

where n_{11} is either 0 or 1 and

$$(1.9) \qquad n_{ij} \geq n_{i,j+1} \geq 0; \quad n_{ij} \geq n_{i+1,j} \geq 0 \ ,$$

For example, $T(2,5)$ and $T(2,6)$ are the number of arrays

$$\begin{array}{ccccc} x & x & x & x & x \\ & & x & x & x \\ & & & & x \ , \end{array} \qquad \begin{array}{cccccc} x & x & x & x & x & x \\ & & x & x & x & x \\ & & & & x & x \ , \end{array}$$

where each x is either 0 or 1 subject only to the conditions (1.9). As a further example, we have $T(2,3) = 5$, the arrays being

$$\begin{array}{ccc} 1 & 1 & 1 \\ & & 1 \ , \end{array} \quad \begin{array}{ccc} 1 & 1 & 1 \\ & & 0 \ , \end{array} \quad \begin{array}{ccc} 1 & 1 & 0 \\ & & 0 \ , \end{array} \quad \begin{array}{ccc} 1 & 0 & 0 \\ & & 0 \ , \end{array} \quad \begin{array}{ccc} 0 & 0 & 0 \\ & & 0 \ . \end{array}$$

Indeed, we find that

$$(1.10) \qquad T(k,n) = F_m(k) \quad (m, k = 1, 2, 3, \cdots) \ .$$

The numbers $N_n(k)$ also occur in connection with triangular arrays of zeros and ones. We prove that

$$(1.11) \qquad N_n(k) = \frac{1}{k} \sum_{j=0}^{k-1} \left[(\rho^{-j} + 1)^k - 1 \right] (\rho^j + 1)^n \ ,$$

$$(1.12) \qquad \sum_{n=0}^{\infty} N_n(k) x^n = \frac{x^k - 1}{x^k - (1 - x)^k} \ ,$$

where ρ denotes a primitive k^{th} root of unity.

Finally, we have included some one-line arrays which can be enumerated in terms of the numbers $F_n(k)$ and $N_n(k)$.

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2. THE NUMBERS $F_n(k)$

For given positive integers m and k, let $T(k,m)$ denote the number of arrays (1.8) subject to the conditions (1.9).

To evaluate $T(k,m)$, we first note that if $n_{1m} = 1$ in (1.8), then $n_{11} = \cdots = n_{1,m-1} = 1$ and there are $T(k,m-k)$ arrangements of the resulting matrix. On the other hand, if $n_{1m} = 0$, then $n_{2m} = \cdots = n_{p+1,m} = 0$ and there are $T(k,m-1)$ arrays possible. This evidently yields

$$T(k,m) = T(k,m-1) + T(k,m-k) \quad (m > k) \quad . \tag{2.1}$$

In the next place, it follows at once from (1.8) and (1.9) that

$$T(k,m) = m + 1 \quad (1 \le m \le k) \ . \tag{2.2}$$

This evidently completes the proof of

<u>Theorem 1.</u> The number of arrays (1.8) subject to the conditions (1.9) is given by

$$T(k,m) = F_m(k) \quad (m,k = 1,2,3,\cdots) \ . \tag{2.3}$$

As an immediate corollary of (2.3) we have

Theorem 2. Let $q_k(n;p)$ denote the number of partitions of n into at most p parts, successive parts differing by at least k. Then

$$\sum_{n=0}^{M} q_k(n;p+1) = F_m(k) \ , \tag{2.4}$$

where $m = kp + r$ $(1 \le r \le k)$ and $M = m(p+1) - k\binom{p+1}{2}$.

Indeed, using the generating function [2]

$$\sum_{n=0}^{\frac{1}{2}m(m+1)} q_1(n;m)x^n = \prod_{j=1}^{m} (1 + x^j) \ ,$$

we easily verify that, for $k = 1$, (2.4) reduces to (1.5). However, Chaundy [1] has noted that, for $k > 1$, the generating function for $q_k(n;p)$ is not known.

3. THE NUMBERS $N_n(k)$

Given positive integers m and k, put $m = pk + r (1 \leqq r \leqq k)$ and, for $1 \leqq j \leqq k$, let $N_j(m,k)$ denote the number of arrays

$$\begin{array}{ccccccc} n_{11} \cdots & n_{1,k+1} \cdots & n_{1,pk+1} \cdots & n_{1m} & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ n_{j1} \cdots & n_{j,k+1} \cdots & n_{j,pk+1} \cdots & n_{jm} & & & \\ & \cdots & \cdots & \cdots & & & \\ & n_{j+k,k+1} \cdots & n_{j+k,pk+1} \cdots & n_{j+k,m} & & & \\ & & \cdots & \cdots & & & \\ & & n_{j+pk,pk+1} \cdots & n_{j+pk,m} & & & \end{array} \tag{3.1}$$

where n_{11} is either 0 or 1 and the n_{ij} are non-negative integers subject to the conditions (1.9). For example, $N_1(5,2)$ and $N_2(5,2)$ are the number of arrays

$$\begin{array}{ccccc} x & x & x & x & x \\ & & x & x & x \\ & & x & x & x \\ & & & & x \\ & & & & x \end{array}, \qquad \begin{array}{ccccc} x & x & x & x & x \\ x & x & x & x & x \\ & & x & x & x \\ & & x & x & x \\ & & & & x \end{array},$$

respectively.

It follows from (3.1) and (1.9) that

$$N_j(m,k) = \binom{m+j}{j} \qquad (1 \leqq m \leqq k), \tag{3.2}$$

$$N_j(m,k) = N_j(m-1,k) + N_{j-1}(m,k) \quad (2 \leqq j \leqq k;\ m > k), \tag{3.3}$$

$$N_1(m,k) = N_1(m-1,k) + N_k(m-k,k) \qquad (m > k). \tag{3.4}$$

The proof of (3.2) is not difficult; (3.3) and (3.4) are proved in exactly the same way as (2.1).

Using (3.3) with $j = k$, we see that

$$N_{k-1}(m,k) = N_k(m,k) - N_k(m - 1,k) ,$$

and, in general,

$$(3.5) \qquad N_{k-j}(m,k) = \sum_{r=0}^{j} (-1)^r \binom{j}{r} N_k(m - r,k) \quad (1 \le j \le k - 1) .$$

Comparing (3.4) and (3.5), we obtain the recurrence

$$(3.6) \qquad N_k(m - k,k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} N_k(m - r,k) ,$$

which should be compared with (1.2).

For $k = 1, 2$ the recurrence (3.6) is easily handled. Indeed, it follows from (3.2) that (3.6) is in agreement with (1.5) and (1.7). Note that (1.7) and (3.5) imply

$$(3.7) \qquad N_1(m,2) = 3 \cdot 2^{m-2} \quad (m \ge 2) .$$

To solve the recurrence (3.6) for general k, we make use of some results from the calculus of finite differences [3]. Let ρ denote a primitive k^{th} root of unity and note that the characteristic polynomial of the recurrence is

$$(x - 1)^k - 1 ,$$

whose roots are $\rho^j - 1$ $(j = 0, 1, 2, \cdots, k - 1)$. Thus there are constants A_0, $A_1, \cdots, A_{k-1}$ such that

$$(3.8) \qquad N_k(n,k) = \sum_{j=0}^{k-1} A_j(\rho^j - 1)^n .$$

We show that

$$A_j = \frac{1}{k}\left[(\rho^{-j}+1)^k - 1\right] \quad (0 \le j \le k-1) \ , \tag{3.9}$$

first noting that we may extend the recurrence (3.6) and define $N_k(0,k) = 1$.

To prove (3.9), we have, for $0 \leqq r \leqq k-1$,

$$\sum_{j=0}^{k-1}\left[(\rho^{-j}+1)^k - 1\right](\rho^j + 1)^r = \sum_{s=0}^{k-1}\binom{k}{s}\sum_{t=0}^{r}\binom{r}{t}\sum_{j=0}^{k-1}\rho^{j(t-s)}$$

$$= k\sum_{s=0}^{r}\binom{k}{s}\binom{s}{t} = k\binom{k+r}{r} \ ,$$

which, using (3.2), implies (3.9).

It follows from (3.8) and (3.9) that

$$N_k(n,k) = \frac{1}{k}\sum_{j=0}^{k-1}\left[(\rho^{-j}+1)^k - 1\right](\rho^j - 1)^n \ , \tag{3.10}$$

so that

$$N_k(n,k) = \sum_{s=0}^{k-1}\binom{k}{s}\sum_{r\equiv s \pmod k}\binom{n}{r}. \tag{3.11}$$

If we define generating functions

$$F_{kj}(x) = \sum_{n=0}^{\infty} N_j(n,k)x^n \quad (1 \le j \le k) \ , \tag{3.12}$$

then it is clear from (3.2) and (3.3) that

$$(1-x)^{j-1}F_{kj}(x) = F_{k1}(x) \quad (j = 2,3,\cdots,k) \ . \tag{3.13}$$

Moreover, using (3.4), we have

$$F_{kk}(x) = x^{-k}(1 - x)F_{k1}(x) - \frac{x^{-k}(1 - x^k)}{1 - x} .$$

Comparison with (3.13) then yields

$$(3.14) \qquad F_{k1}(x) = \frac{(x^k - 1)(1 - x)^{k-1}}{x^k - (1 - x)} ,$$

$$(3.15) \qquad F_{kj}(x) = \frac{(x^k - 1)(1 - x)^{k-j}}{x^k - (1 - x)^k} \qquad (1 \leqq j \leqq k) .$$

We summarize the results of this section by stating

Theorem 3. Let $N_j(n,k)$ denote the number of arrays (3.1) subject to the conditions (1.9). Then $N_j(n,k)$ satisfies (3.6), (3.10), and has generating function (3.15).

4. SOME ONE-LINE ARRAYS

Let $S_k(n_1)$ denote the number of one-line arrays

$$(4.1) \qquad n_1 n_2 n_3 n_4 \cdots ,$$

where the n_j are non-negative integers, subject to the conditions

$$(4.2) \qquad n_j \geqq n_{j+1} + k \quad (j = 1, 2, 3, \cdots) .$$

It is clear from (4.1) and (4.2) that

$$S_k(n) = 1 \qquad (n \le k) ,$$

$$S_k(n) = \sum_{r=0}^{n-k} S_k(r) \qquad (n > k) ,$$

which implies

$$S_k(n) = S_k(n-1) + S_k(n-k) \qquad (n > k) .$$

Thus an easy induction establishes

Theorem 4. The number of arrays (4.1) subject to the conditions (4.2) is given by

$$S_k(n) = 1 \qquad (1 \leqq n \leqq k) , \tag{4.3}$$

$$S_k(n) = F_{n-k}(k) \qquad (n > k) . \tag{4.4}$$

In particular note that (4.3) and (4.4) yield

$$S_2(n) = F_n \qquad (n = 1, 2, 3, \cdots) . \tag{4.5}$$

Returning to the numbers $F_n(k)$, we see from (1.1) and (1.3) that

$$F_{nk+j}(k) - 1 = \sum_{r=0}^{n} F_{rk+j-1}(k) \qquad (1 \leqq j \leqq k) . \tag{4.6}$$

In the next place, for $1 \le j \le k$, let $S_{kj}(n_1)$ denote the number of arrays (4.1), where the n_r are non-negative integers subject to the conditions

$$\begin{aligned} n_r &\geqq n_{r+1} \qquad (r \not\equiv j \pmod{k}) , \\ n_r &> n_{r+1} \qquad (r \equiv j \pmod{k}) . \end{aligned} \tag{4.7}$$

It is immediate from (3.7) that

$$S_{kj}(1) = j \qquad (1 \leqq j \leqq k) , \tag{4.8}$$

$$S_{k,j+1}(n) = 1 + \sum_{r=1}^{n} S_{kj}(r) \qquad (1 \leqq j \leqq k-1) , \tag{4.9}$$

(4.10) $$S_{k1}(n) = 1 + \sum_{r=1}^{n-1} S_{kk}(r) .$$

We shall show that

(4.11) $$S_{kj}(r + 1) = F_{rk+j-1}(k) \quad (1 \le j \le k) .$$

The proof of (4.11) is by induction, the case $r = 0$ being in agreement with (4.8).

Assuming (4.11) for $r \le n - 1$, we see from (4.10) that

$$S_{k1}(n + 1) = F_{(n-1)k}(k) + F_{nk-1}(k) ,$$

which implies

(4.1) $$S_{k1}(n + 1) = F_{nk}(k) .$$

Using (4.6), (4.9), and (4.12), we obtain successively

$$S_{k,j+1}(n + 1) = 1 + \sum_{r=0}^{n} F_{rk+j-1}(k) = F_{nk+j}(k) ,$$

which proves

Theorem 5. The number of arrays (4.1) subject to the conditions (4.7) is given by (4.11).

Finally, we can use the numbers $N_j(n,k)$ to enumerate certain one-line arrays. For $1 \le j \le k$, let $R_{kj}(n)$ denote the number of arrays

(4.13) $$n \; n_1 \; n_2 \; n_3 \; \cdots ,$$

where

(4.14) $$\begin{aligned} n_r &\ge n_{r+1} \quad (r \not\equiv j \pmod k) , \\ n_r &\ge k + n_{r+1} \quad (r \equiv j \pmod k) . \end{aligned}$$

It follows that

$$R_{kj}(n) = \binom{n+j}{j} \qquad (0 \leq n \leq k) \ , \tag{4.15}$$

$$R_{kj}(n) = \sum_{s=0}^{n} R_{k,j-1}(s) \qquad (2 \leq j \leq k) \ , \tag{4.16}$$

$$R_{k1}(n) = \sum_{s=0}^{n-k} R_{kk}(s) \qquad (n > k) \ . \tag{4.17}$$

and we deduce

Theorem 6. The number of arrays (4.13) subject to the conditions (4.14) is given by

$$R_{kj}(n) = N_j(n,k) \qquad (1 \leq j \leq k) \ . \tag{4.18}$$

For convenience of reference, we give the following tables of $F_{n+k}(k)$ and $N_j(n,k)$.

$F_{n+k}(k)$:

k \ n	1	2	3	4	5	6	7
1	4	8	16	32	64	128	256
2	5	8	13	21	34	55	89
3	6	9	13	19	28	41	60
4	7	10	14	19	26	36	50
5	8	11	15	20	26	34	45
6	9	12	16	21	27	34	43
7	10	13	17	22	28	35	43

$N_j(n,k)$:

j	k \ n	1	2	3	4	5	6	7	8
1	1	2	4	8	16	32	64	128	256
1	2	2	3	6	12	24	48	96	192
2	2	3	6	12	24	48	96	192	384
1	3	2	3	4	8	18	38	76	150
2	3	3	6	10	18	36	74	150	300
3	3	4	10	20	38	74	148	298	598
1	4	2	3	4	5	10	25	60	130
2	4	3	6	10	15	25	50	110	240
3	4	4	10	20	35	60	110	220	460
4	4	5	15	35	70	130	240	460	920

5. ADDITIONAL PROPERTIES

The above table of values for $N_j(n,k)$ suggests the formulas

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_{k-j}(n,k) = \begin{cases} 0 & (2 \mid k, \quad n > k), \\ 2N_k(n-k,k) & (2 \nmid k, \quad n > k), \end{cases} \tag{5.1}$$

$$N_{n+r}(n + km - r, k) = N_{n-r}(n + km + r, k) \qquad (n \geq r), \tag{5.2}$$

$$N_r(km + r, k) = 2N_{r-1}(km + r, k) . \tag{5.3}$$

To prove (5.1), we have, using (3.5) and (3.6),

$$\begin{aligned}\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_{k-j}(n,k) &= \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} N_k(n-r,k) \sum_{j=0}^{k-r-1} (-1)^{r+j} \binom{k-r}{j} \\ &= (-1)^{k+1} \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} N_k(n-r,k) \\ &= (1 + (-1)^{k+1}) N_k(n-k,r) ,\end{aligned}$$

which implies (5.1).

In the next place, it follows from (3.5) and (3.10) that

$$(5.4)\qquad N_j(n,k) = \sum_{r=0}^{k-1} \left[(\rho^r + 1)^k - 1\right] (\rho^r + 1)^{n+j-k}\rho^{-jr} \quad (1 \leqq j \leqq k;\ n \geqq k)$$

so that

$$(5.5)\qquad N_{k-j}(n,k) = \sum_{s=0}^{k-1} \binom{k}{s} \sum_{r\equiv s+j \pmod k} \binom{n-j}{r} \quad (1 \leqq j \leqq k).$$

It is clear from (5.4) that

$$\begin{aligned} N_{n+r}(n \pm km - r, k) &= \sum_{s=0}^{k-1} \left[(\rho^s + 1)^{2n+km} - (\rho^s + 1)^{2n+km-r}\right]\rho^{-s(n+r)} \\ &= \sum_{s=0}^{k-1} \left[(\rho^{-s} + 1)^{2n+km} - (\rho^{-s} + 1)^{2n+km-r}\right]\rho^{s(n+r)} \\ &= \sum_{s=0}^{k-1} \left[(\rho^s + 1)^{2n+km} - (\rho^s + 1)^{2n+km-r}\right]\rho^{-s(n-r)} \end{aligned}$$

which completes the proof of (5.2). We remark that (5.3) is an immediate corollary of (5.2).

Note that (5.2) requires only that $n \geqq r$. This follows because (5.4) is valid for all non-negative j.

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