BASES FOR INFINITE INTERVALS OF INTEGERS

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1. INTRODUCTION

In this paper we discuss the problem of representing uniquely each member of an arbitrary infinite interval of integers. The integers of the interval, and no others, are to be expressed as sums of terms of a sequence (b_n) of integers. We also discuss the problem of representing uniquely each positive integer, and no other integer, as the linear combination of terms of a sequence (b_n) of integers, where the coefficients in the linear combination are prescribed and have the value $+1$ or -1. In each problem, roughly speaking, we choose an integer $k \geq 1$ and require that any two terms of (b_n) whose suffixes differ by less than k shall not both be used in the representation of any given integer. The precise definitions and results are in the next section, where we also show the way in which earlier work [1] by one of us (D. E. D.) is related to our definition of an (h,k) base.

In a later paper we will discuss an analogous problem of representing uniquely each real number in the interval $(0,c]$, where c is any positive real number. Finally, we would like to thank Professor R. Rado for his helpful suggestions in the preparation of this paper.

2. STATEMENT OF RESULTS

Throughout this paper, h, k and m are integers such that

$$h + 1 \geq k \geq h \geq 0, \quad k \geq 1 \text{ and } m \geq 1 .$$

Also, unless we state otherwise for a particular sequence, the subscript of the first term of a finite or infinite sequence is the number 1, e.g.,

$$(a_n) = (a_1, a_2, \cdots) \quad .$$

We denote by (v_n) the $(h,k)^{\text{th}}$ Fibonacci sequence defined by

$$(2.1)\qquad \begin{cases} v_n = n & \text{for } 1 \le n \le k, \\ v_n = v_{n-1} + v_{n-k} + (k - h) & \text{for } n > k . \end{cases}$$

An equivalent definition of this sequence was given (for $h \ge 1$) in [1] (p. 144). We denote by (u_n) the $(k,k)^{th}$ Fibonacci sequence and by (f_n) the $(2.2)^{th}$ Fibonacci sequence, which is clearly the original Fibonacci sequence (1, 2, 3, 5, 8, 13, 21, $\cdots$). Further, we write $[a,b]$ for the interval of integers x, $a \le x \le b$, with the obvious interpretation when $a = -\infty$ or $b = +\infty$.

Suppose (a_n), (k_n) is a pair of sequences of positive integers with the following property $\underset{\sim}{P}$.

$\underset{\sim}{P}$. Each integer $N \in [1,\infty]$ has a unique representation

$$N = a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}$$

where $\alpha = \alpha(N)$ and $i_{\nu+1} - i_\nu \ge k$ for $1 \le \nu < \alpha$.

It is shown in [1] (Theorem D) that if (a_n) is increasing and the pair (a_n), (k_n) have the property $\underset{\sim}{P}$ then $k_1 \le k_2 \le k_1 + 1$, $k_2 = k_\nu$ for $\nu \ge 2$, and (a_n) is the $(k_1, k_2)^{th}$ Fibonacci sequence. This result leads us to make the following definition.

<u>Definition 1</u>. A finite or infinite sequence (b_n) of integers is an (h,k) base for an interval $[a,b]$ if each integer $N \in \{0\} \cup [a,b]$ has a unique representation

$$(2.2)\qquad N = b_{i_1} + b_{i_2} + \cdots + b_{i_\alpha} ,$$

where

$$\alpha = \alpha(N),\ i_2 \ge i_1 + h \text{ if } \alpha > 1, \text{ and } i_{\nu+1} \ge i_\nu + k \text{ for } 1 \le \nu < \alpha ,$$

and further, if N is an integer which can be expressed in the form (2.2) then

$$N \in \{0\} \cup [a,b] .$$

Notice that the representation of 0 in the form (2.2) is the empty sum.

Theorem 1 is a statement in this notation of another result proved for $h \ge 1$ in the earlier paper ([1], Theorem C). This result can easily be shown to be true for $h = 0$ also.

<u>Theorem 1</u>. The first n terms $(v_1, v_2, \cdots, v_n)$ of the $(h,k)^{th}$ Fibonacci sequence (v_n) form an (h,k) base for $[1, v_{n+1} - 1]$, and (v_n) forms an (h,k) base for $[1, \infty]$.

Our first new results, Theorems 2-6, are concerned with the existence of (h,k) bases for the infinite interval $[m,\infty]$ with $m \neq 1$, and for the infinite intervals $[-m,\infty]$ and $[-\infty,\infty]$. We conjecture that there is no (h,k) base for $[m,\infty]$ when $m \geq 3$, but have only been able to prove the following theorem.

Theorem 2. If $m > 1$ and (b_n) is an increasing sequence of integers, $b_n \neq (2,3,4,\cdots,2^{n-1},\cdots)$, then (b_n) is not an (h,k) base for $[m,\infty]$. However, $(b_n) = (2,3,4,\cdots,2^{n-1},\cdots)$ is an (h,k) base for $[m,\infty]$ if and only if $h = k = 1$, and $m = 2$. By the statement that (b_n) is an increasing sequence, we mean that $b_1 \leq b_2 \leq \cdots$.

It is easier to deal with the intervals $[-m,\infty]$ and $[-\infty,\infty]$, provided that $h = k$. However, we have been unable to settle the question of the existence of (h,k) bases for these intervals when $h \neq k$.

Theorem 3. If $-m$ is a negative integer then there exists a (k,k) base for $[-m,\infty]$.

For the set of all integers, $[-\infty,\infty]$, there are infinitely many (k,k) bases, and in fact we can choose the sign which each term of a (k,k) base is to have, subject to the condition that the signs change infinitely often.

Theorem 4. Let (s_n) be a sequence such that

$$\left\{\begin{array}{l} s_n \in \{-1,1\} \text{ for } n \geq 1\,, \quad \text{and} \\ s_n \cdot s_{n-1} = -1 \text{ for infinitely many } n > 1\,. \end{array}\right. \tag{2.3}$$

Then there is a (k,k) base (b_n) for $[-\infty,\infty]$ with $s_n b_n > 0$ for $n \geq 1$.

For $k = 2$, we give an explicit example of a (k,k) base for $[-m,\infty]$ in terms of the Fibonacci sequence (f_n). We first represent m in the form

$$m = f_{i_1} + f_{i_2} + \cdots + f_{i_\alpha}\,, \tag{2.4}$$

where

$$i_{\nu+1} \geq i_\nu + 2 \text{ for } 1 \leq \nu < \alpha\,.$$

The existence and uniqueness of this representation is proved by Theorem 1. Next we let (s_n) be the sequence defined in terms of the suffixes i_ν of (2.4) as follows

$$(2.5)\qquad \begin{aligned} s_{i_{\nu+1}} &= -1 \text{ for } 1 \le \nu \le \alpha \\ s_n &= 1 \text{ otherwise.} \end{aligned}$$

Then an explicit formula for a (2,2) base for $[-m,\infty]$ is given in the following theorem.

Theorem 5. Let $-m$ be a negative integer, let the sequence (s_n) be defined as in (2.5), and let

$$(2.6)\qquad b_1 = s_1 \text{ and } b_n = \begin{Bmatrix} s_n f_n & \text{if } s_n \cdot s_{n-1} = 1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{Bmatrix} \text{ for } n > 1 .$$

Then (b_n) is a (2,2) base for $[-m,\infty]$.

Similarly, we have an explicit formula for a (2,2) base for $[-\infty,\infty]$, in terms of the Fibonacci sequence (f_n). We prescribe the sign of each term of the base, subject to the condition that the signs change infinitely often.

Theorem 6. If the sequence (s_n) satisfies (2.3) and the sequence (b_n) is determined in terms of (s_n) by the relations (2.6), then (b_n) is a (2,2) base for $[-\infty,\infty]$ with $s_n b_n \ge 0$ for $n \ge 1$.

So far we have been concerned with unique representations of integers as sums of terms of a base. It is interesting to consider the problem of uniquely representing integers as linear combinations of terms of a sequence (b_n) of integers, where the coefficients in the linear combination are prescribed and have the value $+1$ or -1. We first make the following definition.

Definition 2. Let a sequence $S = (s_n)$, where $s_n \in \{-1,1\}$ for $n \ge 1$, be given. A sequence (b_n) of integers is an $(h+1, k; S)$ base for $[0,\infty]$ if each integer $N \in [0,\infty]$ has a unique representation

$$(2.7)\qquad N = s_\alpha b_{i_1} + s_{\alpha-1} b_{i_2} + \cdots + s_1 b_{i_\alpha} ,$$

where

$$\alpha = \alpha(N), \quad i_2 \ge i_1 + h + 1 \text{ if } \alpha > 1,$$

and

$$i_{\nu+1} \ge i_\nu + k \text{ for } 2 \le \nu < \alpha$$

and further, if N is an integer which can be expressed in the form (2.7) then $N \in [0,\infty]$.

Theorem 1 shows that the $(h,k)^{th}$ Fibonacci sequence (v_n) is an (h,k) base for $[1,\infty]$. It follows that (v_n) is an (h,k) base for the set of all non-negative integers, $[0,\infty]$, and we have been able to determine the conditions under which (v_n) is an $(h+1, k;S)$ base for this same set of integers.

Theorem 7. The $(h,k)^{th}$ Fibonacci sequence (v_n) is an $(h+1, k;S)$ base for $[0,\infty]$ if and only if $s_n = (-1)^{n+1}$ for $n \geq 1$.

In our last theorem we give an explicit formula for the terms of (v_n), the $(h,k)^{th}$ Fibonacci sequence. It is well known that the terms of the Fibonacci sequence (f_n) are sums of the elements in the diagonals of Pascal's triangle, and Theorem 8 extends this result.

Theorem 8.

$$v_n = \sum_{i=k-h}^{\infty} \binom{n-h+(k-1)(2-i)}{i} \quad \text{for } n \geq 1. \tag{2.8}$$

Here, as usual, $\binom{a}{b}$ denotes the binomial coefficient $a!/(a-b)!(b!)$.

3. PROOF OF THEOREM 2

We assume that the sequence (b_n) is increasing and is an (h,k) base for $[m,\infty]$, and in each of the first three cases we deduce a contradiction of definition 1 of an (h,k) base by finding a number which has two representations in the form (2.2).

Lemma 1. $b_n = n + m - 1$ for $1 \leq n \leq m + h$.

Proof. As the sequence (b_n) is increasing, it is strictly increasing, so that $b_1 = m$ and

$$b_n \geq m + n - 1 \quad \text{for } n \geq 1. \tag{3.1}$$

The smallest number of the form (2.2) with $\alpha > 1$ is $b_1 + b_{1+h}$, and, by (3.1), $b_1 + b_{1+h} \geq 2m + h$. Hence $b_n = m + n - 1$ for all $n \geq 1$ such that $m + n - 1 < 2m + h$; i.e., $n \leq m + h$. This proves Lemma 1.

We consider now the various cases.

Case [1]. $m \geq 3$. Then by Lemma 1,

$$b_1 + b_{h+3} = m + (m + h + 3 - 1) = (m + 1) + (m + h + 2 - 1) = b_2 + b_{h+2} .$$

Case [2]. $m = 2$, $k > 1$. By Lemma 1, $b_n = n + 1$ for $1 \leq n \leq h + 2$, and so $b_1 + b_{1+h} = 4 + h$, $b_1 + b_{2+h} = 5 + h$, and $b_2 + b_{2+h} = 6 + h$. Clearly, $6 + h$ is the largest number which can be represented in the form (2.2) with $i_\alpha \leq 2 + h$ and $\alpha = 2$. However, the smallest number which can be represented with $\alpha = 3$ is

$$b_1 + b_{1+h} + b_{1+h+k} \geq 4 + h + b_{3+h} > 4 + h + 6 + h \geq 10 + h .$$

Therefore $b_{3+h} = 7 + h$. But $b_1 + b_{3+h} = 2 + (7 + h) = 9 + h$, so that $8+h$ has no representation with $i_\alpha \leq 3 + h$. Hence $b_{4+h} = 8 + h$. But then we have

$$b_1 + b_{4+h} = 2 + (8 + h) = 3 + (7 + h) = b_2 + b_{3+h} .$$

Case [3]. $m = 2$, $k = 1$, $h = 0$. Then by Lemma 1, $b_1 = 2$ and $b_2 = 3$. Therefore the representations of 4, 5, 6 and 7 are $b_1 + b_1$, $b_1 + b_2$, $b_2 + b_2$ and $b_1 + b_1 + b_2$ respectively. The number 8 cannot be represented in the form (2.2) with $i_\alpha \leq 2$. Hence $b_3 = 8$. Similarly the number 9 cannot be represented with $i_\alpha \leq 3$. Hence $b_4 = 9$. But then $b_1 + b_4 = 2 + 9 = 3 + 8 = b_2 + b_3$.

We have now only to deal with the cases when $m = 2$, $k = 1$, $h = 1$. It follows, therefore, from the contradictions obtained in the first 3 cases that if (b_n) is an (h,k) base for $[m,\infty]$, then $h = k = 1$ and $m = 2$.

Case [4]. $h = k = 1$, $m = b_1 = 2$, $b_2 = 3$ and $b_n = 2^{n-1}$ for $n \geq 3$. In this case (b_n) is a $(1,1)$ base for $[2,\infty]$.

For let $N \geq 2$ be an integer. If N is even, then its representation in the form (2.2) is the binary representation, which is unique. If N is odd, then $N - 3$ is even, and so the representation of N is the binary representation of $N - 3$ together with b_2; hence this representation is also unique.

Notice that, for $p \ge 3$, each of the numbers $2, 3, \cdots, 2^{p-1} - 2, 2^{p-1} - 1, 2^{p-1} + 1$, and no others, can be represented in the form (2.2) using $(b_1, b_2, \cdots, b_{p-1})$. This fact is used in the proof of the next case.

Case [5]. $h = k = 1$, $m = 2$, $(b_n \ne (2, 3, 4, 8, \cdots, 2^{n-1}, \cdots)$. Again we assume that (b_n) is increasing and is an (h,k) base for $[m,\infty]$, so that, by Lemma 1, $b_1 = 2$ and $b_2 = 3$. Let $p \ge 3$ be an integer. Suppose that $b_p \ne 2^{p-1}$, and, if $p \ge 3$, also suppose that $b_3 = 4$, $b_4 = 8, \cdots, b_{p-1} = 2^{p-2}$. Then, by the remark at the end of the last case, $b_p \ge 2^{p-1} + 1$. But then 2^{p-1} has no representation in the form (2.2), which contradicts definition 1 of an (h,k) base. This completes the proof of Theorem 2.

4. PROOFS OF THEOREMS 3, 4, 5 and 6

Throughout this section, namely Lemmas 2-8 and the proofs of Theorems 3-6, the sequences (t_n), (a_n), (d_n) and (e_n) are as defined immediately below. We let (t_n) be a sequence such that $t_n \in \{-1, 1\}$ for $n \ge 1$. The three sequences (a_n), (d_n) and (e_n) are simultaneously defined by induction in terms of the sequence (t_n). First we put $a_1 = t_1$ and $d_n = e_n = 0$ for $n \le 0$. If $n > 1$ and we have defined the terms d_ν, e_ν for $\nu \le n - 2$, and the terms a_ν for $1 \le \nu \le n - 1$, then we define d_{n-1}, e_{n-1} and a_n as follows.

i) d_{n-1} is the largest, and e_{n-1} is the smallest of the number 0 and the numbers representable in the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}, \tag{4.1}$$

where

$$i_\alpha \le n - 1 \quad \text{and} \quad i_{\nu+1} \ge i_\nu + k \quad \text{for} \quad 1 \le \nu \le \alpha .$$

ii)

$$a_n = \begin{cases} d_{n-1} - e_{n-k} + 1 & \text{if } t_n = +1 \\ e_{n-1} - d_{n-k} - 1 & \text{if } t_n = -1 \end{cases} . \tag{4.2}$$

The relation (4.2) is clearly true for $n = 1$ also.

Lemma 2. (i) For all n, $0 \le d_{n-1} \le d_n$ and $e_n \le e_{n-1} \le 0$.

(ii) For $n \ge 1$, $t_n a_n \ge 0$.

Proof. (i) Follows immediately from the definitions of (d_n) and (e_n).

(ii) For $n \geq 1$, if $t_n = +1$, then, by (4.2) and part (i),

$$a_n = d_{n-1} - e_{n-k} + 1 \geq 1 .$$

The proof when $t_n = -1$ is similar and completes the proof of Lemma 2.

Lemma 3. For $n \geq 1$, if $t_n = +1$, then $d_n = d_{n-k} + a_n$ and $e_n = e_{n-1}$, and if $t_n = -1$, then $e_n = e_{n-k} + a_n$ and $d_n = d_{n-1}$.

Proof. (i) We assume that $t_n = +1$ and show that $e_n = e_{n-1}$. Since $t_n = +1$, by Lemma 2(ii), $a_n > 0$. The number e_{n-1} is, by definition, the smallest of the number 0 and the numbers representable in the form (4.1), and since $a_n > 0$, no smaller number can be formed by adding a_n. Hence $e_n = e_{n-1}$. Similar reasoning shows that $d_n = d_{n-1}$ if $t_n = -1$.

(ii) We assume that $t_n = +1$ and show that $d_n = d_{n-k} + a_n$. From the definition of (d_n), $d_n \geq d_{n-k} + a_n$. We suppose that $d_n > d_{n-k} + a_n$, so that $d_n = d_{n-r-k} + a_{n-r}$ for some $r > 0$. Hence $d_{n-r-k} + a_{n-r} > d_{n-k} + a_n$. However by Lemma 2(i), $d_{n-r-k} \leq d_{n-k}$, so that $a_{n-r} \geq a_n$. Since $t_n = +1$ it follows from Lemma 2(ii) that $a_n > 0$. Therefore $a_{n-r} > 0$ and so $t_{n-r} = +1$. Therefore, by (4.2),

$$d_{n-r-1} - a_{n-r-k} + 1 > d_{n-1} - e_{n-k} + 1 . \tag{4.3}$$

However, by Lemma 2(i), $d_{n-1} \geq d_{n-r-1}$ and $-e_{n-k} \geq -e_{n-r-k}$, which contradicts (4.3) and so proves that $d_n = d_{n-k} + a_n$. The proof that if $t_n = -1$ then $e_n = e_{n-k} + a_n$ is similar. This completes the proof of Lemma 3.

Lemma 4. For all n, the finite sequence $(a_1, a_2, \cdots, a_n)$ is a (k,k) base for $[e_n, d_n]$.

Proof. We use induction upon n. When $n < 1$, $(a_1, a_2, \cdots, a_n) = \phi$, the empty set. Since $e_n = d_n = 0$, the lemma is true in this case.

Let $m \geq 1$, and suppose the lemma is true for $n < m$. Then $(a_1, a_2, \cdots, a_{m-k})$ is a (k,k) base for $[e_{m-k}, d_{m-k}]$. From (4.2) and Lemma 2(i), if $t_m = +1$ then $a_m + e_{m-k} > d_{m-k}$, and if $t_m = -1$ then $a_m + d_{m-k} < e_{m-k}$. Therefore $(a_1, a_2, \cdots, a_{m-k}, a_m)$ is a (k,k) base for

$$[e_{m-k}, d_{m-k}] \cup [e_{m-k} + a_m, d_{m-k} + a_m].$$

Also by the induction hypothesis, $(a_1, a_2, \cdots, a_{m-1})$ is a base for $[e_{m-1}, d_{m-1}]$. By (4.2), if $t_m = +1$ then $d_{m-1} + 1 = a_m + e_{m-k}$, and if $t_m = -1$ then $e_{m-1} - 1 = a_m + d_{m-k}$. Since also, from Lemma 2(i),

$$[e_{m-1}, d_{m-1}] \supseteq [e_{m-k}, d_{m-k}]$$

it follows that $(a_1, a_2, \cdots, a_m)$ is a (k,k) base for $[e_{m-k}, d_{m-k} + a_m]$ if $t_m = +1$, or for $[e_{m-k} + a_m, d_{m-1}]$ if $t_m = -1$. Hence, by Lemma 3, $(a_1, a_2, \cdots, a_m)$ is a (k,k) base for $[e_m, d_m]$.

Lemma 4 now follows by induction.

Proof of Theorem 4. Suppose $t_n = s_n$ for $n \geq 1$, where (s_n) is the sequence defined in (2.3). Then (t_n) has the additional property that $t_n \cdot t_{n-1} = -1$ for infinitely many $n > 1$. It is then clear from Lemmas 2(ii) and 3 that $d_n \to \infty$ and $e_n \to -\infty$ as $n \to \infty$, so that, by Lemma 4 (a_n) is a (k,k) base for $[-\infty, \infty]$. We have already shown (Lemma 2(ii)) that $a_n t_n > 0$ for $n \geq 1$, so that Theorem 4 is proved.

Only part (i) of the following lemma is needed in this section. Part (ii) is used in Section 5. We let $N_n(\ell)$ be the number of finite sequences $(i_1, i_2, \cdots, i_\alpha)$ of positive integers such that

(4.4) $1 \leq i_\alpha \leq n, \quad i_2 \geq i_1 + \ell \quad \text{if } \alpha > 1, \text{ and } i_{\nu+1} \geq i_\nu + k \text{ for } 2 \leq \nu < \alpha;$

and are only interested in the values $\ell = h$ and $\ell = h + 1$.

Lemma 5. (i) For $n \geq 1$, $N_n(h) = v_{n+1}$,

(ii) For $n \geq 1$, $N_n(h + 1) = v_n + 1$.

Proof. (i) By Theorem 1, for $n \geq 1$, there is a 1:1 correspondence between sums of the form $v_{i_1} + v_{i_2} + \cdots + v_{i_\alpha}$ with condition (4.4) applied with $\ell = h$, and the integers in $[0, v_{n+1} - 1]$. Hence $N_n(h) = v_{n+1}$.

(ii) If each finite sequence $(i_1, i_2, \cdots, i_\alpha)$ of positive integers, with condition (4.4) applied with $\ell = h$, is transformed by putting $i_1 = j_1$ and $i_\nu + 1 = j_\nu$ for $2 \leq \nu \leq \alpha$, then we obtain all but one of the finite sequences $(j_1, j_2, \cdots, j_\alpha)$ of positive integers, where $1 \leq j_\alpha \leq n + 1$, $j_2 \geq j_1 + h + 1$ if $\alpha \geq 1$ and $j_{\nu+1} \geq j_\nu + k$ for $2 \leq \nu < \alpha$. The finite sequence we do not obtain is (j_1), when $j_1 = n + 1$. Therefore, by part (i), $N_{n+1}(h + 1) = v_{n+1} + 1$ for $n \geq 1$. Hence $N_n(h + 1) = v_n + 1$ for $n \geq 2$. As part (ii) is clearly true when $n = 1$, the proof of Lemma 5 is completed.

Lemma 6.

$$(d_n - d_{n-1}) - (e_n - e_{n-1}) = \begin{cases} u_{n-k+1} & \text{for } n \geq k \\ 1 & \text{for } 1 \leq n < k \\ 0 & \text{for } n < 1 . \end{cases}$$

The sequence (u_n) is the $(k,k)^{\text{th}}$ Fibonacci sequence.

Proof. By Lemma 5(i), $N_n(k) = u_{n+1}$ for $n \geq 1$. Therefore it follows from Lemma 4 that $d_n - e_n = u_{n+1} - 1$. Hence for $n \geq 2$,

$$\begin{aligned}(d_n - d_{n-1}) - (e_n - e_{n-1}) &= (d_n - e_n) - (d_{n-1} - e_{n-1}) \\ &= (u_{n+1} - 1) - (u_n - 1) \\ &= \begin{cases} u_{n-k+1} & \text{for } n \geq k , \\ 1 & \text{for } 2 \leq n < k . \end{cases}\end{aligned}$$

The result is easily seen to be true for $n = 1$ and is trivially true for $n < 1$. This proves Lemma 6.

Lemma 7. If $k = 2$, then $a_1 = t_1$ and for $n > 1$,

$$a_n = \begin{cases} t_n f_n & \text{if } t_n \cdot t_{n-1} = 1 , \\ t_n f_{n-1} & \text{if } t_n \cdot t_{n-1} = -1 . \end{cases}$$

The sequence (f_n) is the $(2,2)^{\text{th}}$ Fibonacci sequence $(1, 2, 3, 5, 8, \cdots)$.

Proof. By definition $a_1 = t_1$.

Let $n \geq 2$ and $t_n \cdot t_{n-1} = 1$. By Lemma 3, if $t_1 = +1$ then $a_n = d_n - d_{n-2}$ and $e_n = e_{n-1} = e_{n-2}$, and if $t_n = -1$ then $a_n = e_n - e_{n-2}$ and $d_n = d_{n-1} = d_{n-2}$. Also, by Lemma 2(i), $0 \leq d_{n-2} \leq d_n$ and $e_n \leq e_{n-2} \leq 0$. Hence

$$\begin{aligned} a_n &= \{t_n (d_n - d_{n-2}) - (e_n - e_{n-2})\} \\ &= t_n\{(d_n - d_{n-1}) - (e_n - e_{n-1}) + (d_{n-1} - d_{n-2}) - (e_{n-1} - e_{n-2})\} \\ &= \begin{Bmatrix} t_n(f_{n-1} + f_{n-2}) & \text{if } n \geq 3 \\ t_n(f_{n-1} + 1) & \text{if } n = 2 \end{Bmatrix}, \text{ by Lemma 6,} \\ &= t_n f_n . \end{aligned}$$

Now let $t_n \cdot t_{n-1} = -1$. Then similarly by Lemmas 2(i) and 3,

$$\begin{aligned} a_n &= \begin{cases} d_n - d_{n-1} & \text{if } t_n = +1 , \\ e_n - e_{n-1} & \text{if } t_n = -1 , \end{cases} \\ &= t_n\{(d_n - d_{n-1}) - (e_n - e_{n-1})\} \\ &= t_n f_{n-1}, \text{ by Lemma 6 .} \end{aligned}$$

This proves Lemma 7.

Proof of Theorem 6. We take $k = 2$. We suppose that $t_n = s_n$ for $n \geq 1$, where (s_n) is the sequence defined in (2.3). Then, by Theorem 4, (a_n) is a (2,2) base for $[-\infty, \infty]$ with $a_n s_n > 0$ for $n \geq 1$. But by Lemma 7, $a_1 = s_1$ and

$$a_n = \begin{Bmatrix} s_n f_n & \text{if } s_n \cdot s_{n-1} = 1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{Bmatrix} \text{ for } n \geq 1.$$

Therefore the sequence (a_n) is the same as the sequence (b_n) defined in the statement of Theorem 6. Hence (b_n) is a (2,2) base for $[-\infty,\infty]$ with $a_n b_n > 0$ for $n \geq 1$. This proves Theorem 6.

Lemma 8. For $n \geq 1$, if x is an integer such that $-u_{n+1} + 1 \leq x \leq 0$ then there exists a choice of $(t_1, t_2, \cdots, t_n)$ for which $e_n = x$.

Proof. We use induction upon n. If $t_1 = +1$ then $e_1 = 0$, while if $t_1 = -1$ then $e_1 = -1$; since $-u_2 + 1 = -1$, the Lemma is true in the case when $n = 1$.

Let $m \geq 2$ be an integer and suppose that the Lemma is true for $1 \leq n < m$. Then if $-u_m + 1 \leq x \leq 0$ there exists a choice of $(t_1, t_2, \cdots, t_{m-1})$, which we denote by $(t_1', t_2', \cdots, t_{m-1}')$, for which $e_{m-1} = x$. Hence, if we choose $(t_1, t_2, \cdots, t_m)$ to be $(t_1', t_2', \cdots, t_{m-1}', +1)$, then by Lemma 3, $e_m = x$.

However, suppose that

$$(4.5) \qquad -u_{m+1} \leq x \leq -u_m + 1 .$$

Then

$$(4.6) \qquad x + u_m \leq 0 .$$

Therefore, by (4.5) and (4.6),

$$(4.7)\quad \begin{cases} -u_{m+1} + 1 + u_{m+1-k} \le x + u_{m+1-k} \le 0 \text{ if } m \ge k\,, \\ \text{and} \\ -u_{m+1} + 1 + 1 \le x + 1 \le 0\,. \end{cases}$$

But, from (2.1),

$$(4.8)\qquad -u_m + 1 = \begin{cases} -u_{m+1} + 1 + u_{m+1-k} & \text{if } m \ge k\,, \\ -u_{m+1} + 1 + 1 & \text{if } 2 \le m < k\,. \end{cases}$$

Therefore, by (4.7) and (4.8),

$$(4.9)\qquad \begin{cases} (-u_m + 1) \le x + u_{m+1-k} \le 0 & \text{if } m \ge k, \text{ and} \\ (-u_m + 1) \le x + 1 \le 0 & \text{if } 2 \le m \le k\,. \end{cases}$$

Therefore, by the induction hypothesis and (4.9), there exists a choice of $(t_1, t_2, \cdots, t_{m-1})$, which we denote by $(t_1', t_2', \cdots, t_{m-1}')$, for which

$$(4.10)\qquad e_{m-1} = \begin{cases} x + u_{m+1-k} & \text{if } m \ge k\,; \\ x + 1 & \text{if } 2 \le m < k\,. \end{cases}$$

If we choose $(t_1, t_2, \cdots, t_m)$ to be $(t_1', t_2', \cdots, t_{m-1}', -1)$, then by Lemma 3, $d_m = d_{m-1}$, and so by Lemma 6,

$$(4.11)\qquad e_m = \begin{cases} e_{m-1} - u_{m+1-k} & \text{if } m \ge k\,, \\ e_{m-1} - 1 & \text{if } 2 \le m < k\,. \end{cases}$$

Hence, by (4.10) and (4.11), $e_m = x$.

Lemma 8 now follows by induction.

Proof of Theorem 3. If p is an integer such that $-u_{p+1} + 1 \le -m$ then, by Lemmas 4 and 8, there exists a choice of $(t_1, t_2, \cdots, t_p)$, which we denote by $(t_1', t_2', \cdots, t_p')$, such that $(a_1, a_2, \cdots, a_p)$ is a (k,k) base for $[-m, d_p]$.

Then, if $t_n = t'_n$ for $1 \le n \le p$ and $t_n = +1$ for $n > p$, by Lemmas 3 and 4, $(a_1, a_2, \cdots, a_n)$ is a (k,k) base for $[-m, d_n]$ for $n \ge p$. By Lemmas 2(ii) and 3, $d_n \to \infty$ as $n \to \infty$. Hence (a_n) is a (k,k) base for $[-m, \infty]$.

Proof of Theorem 5. We take $k = 2$. We suppose that $t_n = s_n$ for $n \ge 1$, where (s_n) is the sequence defined in (2.5). By (2.4), $i_{\nu+1} \ge i_\nu + 2$ for $1 \le \nu < \alpha$, and so, by (2.5) and Lemma 2(ii), for $n \ge i_\alpha + 1$,

$$e_n = a_{i_1+1} + a_{i_2+1} + \cdots + a_{i_\alpha+1} . \tag{4.12}$$

However, by Lemma 7, $a_1 = s_1$ and

$$a_n = \begin{cases} s_n f_n & \text{if } s_n \cdot s_{n-1} = +1 \\ s_n f_{n-1} & \text{if } s_n \cdot s_{n-1} = -1 \end{cases} \quad \text{for } n \ge 2 \tag{4.13}$$

Hence the sequence (a_n) is the same as the sequence (b_n) defined in (2.6). But by (4.13) and (2.5), $a_{i_\nu+1} = s_{i_\nu+1} f_{i_\nu} = -f_{i_\nu}$ for $1 \le \nu \le \alpha$. Hence, by (4.12) and (2.4), $e_n = -m$ for $n \ge i_\alpha + 1$. From Lemmas 2(ii) and 3, $d_n \to \infty$ as $n \to \infty$, and so, by Lemma 4, (b_n) is a (2,2) base for $[-m, \infty]$.

5. PROOF OF THEOREM 7

Let $S = (s_n)$ be a sequence such that $s_n \in \{-1, 1\}$ for $n \ge 1$, let m be a positive integer, and let $(i_1, i_2, \cdots, i_\alpha)$ be a finite sequence of positive integers such that

$$i_2 \ge i_1 + h + 1 \text{ if } \alpha > 1 \text{ and } i_{\nu+1} \ge i_\nu + k \text{ for } 2 \le \nu < \alpha . \tag{5.1}$$

Lemma 9. If $(i_1, i_2, \cdots, i_\alpha) = \phi$, the empty set, then

$$s_i v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = 0 .$$

Lemma 10. If $s_1 = +1$ then $s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \ge 0$.

Proof. Let $s_1 = 1$. If $\alpha = 1$, then $s_i v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = v_{i_\alpha} \ge 1$. If $\alpha > 1$ then

$$
\begin{aligned}
s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} &\geq v_{i_\alpha} - (v_{i_{\alpha-1}} + v_{i_{\alpha-2}} + \cdots + v_{i_1}) \\
&\geq v_{i_\alpha} - (v_{i_{\alpha-1}+1} - 1), \text{ by Theorem 1,} \\
&\geq 1.
\end{aligned}
$$

This, together with Lemma 9, proves Lemma 10.

Proof of Sufficiency. Suppose that $s_n = (-1)^{n+1}$ for $n \geq 1$, and that $i_\alpha \leq m$. Since (v_n) is a strictly increasing sequence, it follows that if $\alpha \geq 1$ then $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \leq v_{i_\alpha} \leq v_m$. Hence, and in view of Lemmas 9 and 10,

$$
0 \leq s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} \leq v_m . \tag{5.2}
$$

We show now that any two distinct finite sequences $(i_1, i_2, \cdots, i_\alpha)$ which satisfy (5.1) yield distinct values of $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$. Suppose therefore that two such distinct finite sequences are $(j_1, j_2, \cdots, j_\beta)$ and $(g_1, g_2, \cdots, g_\gamma)$. We suppose without loss of generality that $v_{j_\beta} \geq v_{g_\gamma}$, and consider three cases.

Case [1]. $\beta = 1$. Then

$$
s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} = v_{j_\beta} \geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1}.
$$

Case [2]. $\beta = 2$. Then

$$
\begin{aligned}
s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} &= v_{j_\beta} - v_{j_{\beta-1}} \\
&\geq v_{j_\beta} - v_{j_\beta - h - 1} \\
&\left.\begin{cases} = v_{j_\beta} - v_{j_\beta - k} \geq v_{j_\beta - 1}, & \text{if } k = h+1 \\ \geq v_{j_\beta} - v_{j_\beta - k} = v_{j_\beta - 1}, & \text{if } k = h \end{cases}\right\} \text{by (2.1),} \\
&\geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1} .
\end{aligned}
$$

Case [3]. $\beta = 2$.
Then

$$\begin{aligned}
s_1v_{j_\beta} + s_2v_{j_{\beta-1}} + \cdots + s_\beta v_{j_1} &> v_{j_\beta} - v_{j_{\beta-1}} \\
&\geq v_{j_\beta} - v_{j_{\beta-k}} \\
&\geq v_{j_\beta} - v_{j_\beta\text{-}k} - (k - h) \\
&= v_{j_\beta} - 1, \quad \text{by } (2.1)\ , \\
&\geq v_{g_\gamma} \geq s_1v_{g_\gamma} + s_2v_{g_{\gamma-1}} + \cdots + s_\gamma v_{g_1} .
\end{aligned}$$

By Lemma 5(ii), the number of distinct finite sequences $(i_1, i_2, \cdots, i_\alpha)$ with $i_\alpha \leq m$ which satisfy (5.11) is $v_m + 1$. Therefore, since any two such distinct finite sequences yield distinct values of $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$, and in view of (5.2), it follows that $(v_1, v_2, \cdots, v_m)$ is an $(h + 1, k; S)$ base for $[0, v_m]$ when $s_n = (-1)^{n+1}$ for $n \geq 1$. The sufficiency of the condition follows.

Proof of Necessity. Suppose that (v_n) is an $(h + 1, k; S)$ base for $[0, \infty]$. We show that $s_n = (-1)^{n+1}$ for $n \geq 1$. Clearly $s_1 = +1$, for otherwise $s_1v_1 = -1$, a contradiction. We suppose that $s_n = (-1)^{n+1}$ for $1 \leq n \leq m$ and that $s_{m+1} = (-1)^{m+1}$, and deduce a contradiction in every case.

Case [1]. $m = 1$. Then $s_1 = s_2 = +1$. We write $M = v_{h+2} - v_1$. If $\alpha = 1$ and $i_\alpha \geq h + 2$ then $s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} = v_{i_\alpha} > M$, whereas if $\alpha = 1$ and $i_\alpha \leq h + 1$ then

$$\begin{aligned}
M &= v_{h+2} - v_1 \\
&= v_{h+1} + v_{h+2-k} + (k - h) - v_1, \quad \text{by } (2.1) \\
&= \begin{cases} v_{h+1} + v_2 - v_1, & \text{if } h = k\ , \\ v_{h+1} + v_1 + 1 - v_1, & \text{if } h + 1 = k, \end{cases} \\
&> v_{h+1} \geq v_{i_\alpha} = s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} .
\end{aligned}$$

On the other hand if $\alpha > 1$ then $i_\alpha \geq h + 2$, and so

$$\begin{aligned}
s_1v_{i_\alpha} + s_2v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} &\geq v_{i_\alpha} + v_{i_{\alpha-1}} - (v_{i_{\alpha-2}} + v_{i_{\alpha-3}} + \cdots + v_{i_1}) \\
&\geq v_{i_\alpha} + v_{i_{\alpha-1}} - (v_{i_{\alpha-2}+1} - 1), \\
&\qquad \text{by Theorem 1,} \\
&\geq v_{i_\alpha} + 1 \\
&> v_{h+2} > M .
\end{aligned}$$

Hence $M \neq s_1 v_{i_\alpha} + s_2 v_{i_\alpha} + \cdots + s_\alpha v_{i_1}$ for any finite sequence $(i_1, i_2, \cdots, i_\alpha)$ satisfying (5.1), which contradicts our assumption that (v_n) is an $(h+1, k; S)$ base for $[0,\infty]$.

Case [2]. $m > 1$. We write

$$(5.3) \qquad N = s_1 v_{(m-1)k+h+2} + s_2 v_{(m-2)k+h+2} + \cdots + s_m v_{h+2} + s_{m+1} v_1 .$$

It follows from Lemma 10 that $N \geq 0$. If $m = 2$ then $N = v_{k+h-2} - v_{h+2} - v_1 < v_{k+h+2} = v_{(m-1)k+h+2}$, while if $m > 2$ then

$$\begin{aligned} N &\leq v_{(m-1)k+h+2} - (v_{(m-2)k+h+2} - v_{(m-3)k+h+2}) + 1 \\ &= v_{(m-1)k+h+2} - (v_{(m-2)k+h+1} + (k-h)) + 1, \quad \text{by (2.1)}, \\ &< v_{(m-1)k+h+2} . \end{aligned}$$

Hence

$$0 \leq N \leq v_{(m-1)k+h+2} .$$

Now N is the only number of the form $s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$ with $\alpha = m+1$ and $i_\alpha \leq (m-1)k + h + 2$. Hence, by the proof of the sufficiency,

$$\{n: n = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1} ;\ i_\alpha \leq (m-1)k + h + 2 \text{ and } \alpha \leq m\} \cup \{N - 2s_{m+1} v_1\} = \{0, 1, 2, \cdots, v_{(m-1)k+h+2}\} .$$

Therefore, by (5.4), N can be put in the form

$$N = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$$

with $\alpha \leq m$. Hence, and by (5.3) N has two representations in the form $N = s_1 v_{i_\alpha} + s_2 v_{i_{\alpha-1}} + \cdots + s_\alpha v_{i_1}$, which contradicts our assumption that (v_n) is an $(h+1, k; S)$ base for $[0,\infty]$.

We conclude therefore that $s_n = (-1)^{n+1}$ for $n \geq 1$. This completes the proof of Theorem 7.

6. PROOF OF THEOREM 8

We show that if (v_n) is defined by (2.8) then the defining relations (2.1) of the $(h,k)^{th}$ Fibonacci sequence hold.

If $a < b$ then $\binom{a}{b} = 0$. Hence the infinite sum of (2.8) contains only a finite number of non-zero terms. In fact, for $1 \leq n \leq k$, the relation (2.8) reduces to

$$v_n = \binom{n+k-2}{0} + \binom{n-1}{1}$$

if $k = h$, or $v_n = \binom{n}{1}$ if $k = h + 1$, and so the first of the relations (2.1) holds. On the other hand, if $n > k$, by checking each stage with $h = k$ and $h + 1 = k$, and using the fact that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1},$$

we have

$$\begin{aligned}
v_{n-1} &+ v_{n-k} + (k-h) \\
&= (k-h) + \sum_{i=k-h}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i} \\
&\qquad + \sum_{i=k-h}^{\infty} \binom{n-k-h+(k-1)(2-i)}{i} \\
&= 1 + \sum_{i=1}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i} + \sum_{i=1+k-h}^{\infty} \binom{n-1-h+(k-1)(2-i)}{i-1} \\
&= 1 - (k-h) + \sum_{i=1}^{\infty} \left\{ \binom{n-1-h+(k-1)(2-i)}{i} \right. \\
&\qquad \left. + \binom{n-1-h+(k-1)(2-i)}{i-1} \right\}
\end{aligned}$$

$$= \sum_{i=k-h}^{\infty} \binom{n - h + (k - 1)(2 - i)}{i}$$

$$= v_n \text{ , as required.}$$

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A NEW IMPORTANT FORMULA FOR LUCAS NUMBERS

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The formula

$$\frac{L_{10n}}{L_{2n}} = (L_{4n} - 3)^2 + (5F_{2n})^2 \tag{1}$$

may be easily verified putting $L_n = \alpha^n + \beta^n$,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} , \qquad \alpha\beta = -1 ,$$

Since for $n > 0$, (1) gives a decomposition of L_{10n}/L_{2n} into a sum of 2 squares, and since any divisor of a sum of 2 squares is -1 (mod 4), it follows that any primitive divisor of L_{10n}, $n > 0$, is -1 (mod 4).

* * * * *