FIBONACCI NUMBERS AND GENERALIZED BINOMIAL COEFFICIENTS

V. E. HOGGATT, JR.
San Jose State College, San Jose, California

DEDICATED TO THE MEMORY OF MARK FEINBERG

1. INTRODUCTION

The first time most students meet the binomial coefficients is in the expansion

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j , \quad n \geq 0 ,$$

where

$$\binom{n}{m} = 0 \text{ for } m > n , \qquad \binom{n}{n} = \binom{n}{0} = 1, \text{ and}$$

(1) $$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad 0 < m < n$$

Consistent with the above definition is

(2) $$\binom{n}{m} = \frac{n(n-1)\cdots 2 \cdot 1}{m(m-1)\cdots 2 \cdot 1 (n-m)(n-m-1)\cdots 2 \cdot 1} = \frac{n!}{m!\,(n-m)!} ,$$

where

$$n! = n(n-1)\cdots 2 \cdot 1 \text{ and } 0! = 1 .$$

Given the first lines of Pascal's arithmetic triangle one can extend the table to the next line by using directly definition (2) or the recurrence relation (1).

(Received Oct., 1965)

We now can see just how the ordinary binomial coefficients $\binom{n}{m}$ are related to the sequence of integers $1, 2, 3, \cdots, k, \cdots$. Let us generalize this observation using the Fibonacci sequence.

II. THE FIBONOMIAL COEFFICIENTS

Let the Fibonomial coefficients (which are a special case of the generalized binomial coefficients) be defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_2 F_1}{(F_m F_{m-1} \cdots F_2 F_1)(F_{m-n} F_{m-n-1} \cdots F_2 F_1)} , \quad 0 < m < n ,$$

and

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 ,$$

where F_n is the n^{th} Fibonacci number, defined by

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1 .$$

We next seek a convenient recurrence relation, like (1) for the ordinary binomial coefficients, to get the next line from the first few lines of the Fibonomial triangle, the generalization of which will come shortly.

To find two such recurrence relations we recall the Q-matrix,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ,$$

for which it is easily established by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad n \geq 0 .$$

The Laws of Exponents hold for the Q-matrix so that

$$Q^n = Q^m Q^{n-m} .$$

Thus

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}\begin{pmatrix} F_{n-m+1} & F_{n-m} \\ F_{n-m} & F_{n-m-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_{m+1}F_{n-m+1} + F_mF_{n-m} & F_{m+1}F_{n-m} + F_mF_{n-m-1} \\ F_mF_{n-m+1} + F_{m-1}F_{n-m} & F_mF_{n-m} + F_{m-1}F_{n-m-1} \end{pmatrix}$$

yielding, upon equating corresponding elements,

(A) $$F_n = F_{m+1}F_{n-m} + F_mF_{n-m-1} \quad \text{(upper right)} ,$$

(B) $$F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m} \quad \text{(lower left)} .$$

These two identities will be very handy in what follows.

Define C so that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_nF_{n-1} \cdots F_2F_1}{(F_mF_{m-1} \cdots F_2F_1)(F_{n-m}F_{n-m-1} \cdots F_2F_1)} = F_nC .$$

With C defined above, then

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m}C \text{ and } \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_m C .$$

Returning now to identity (A),

$$F_n = F_{m+1}F_{n-m} + F_m F_{n-m-1} ,$$

we may write for $C \neq 0$,

$$F_n C = F_{m+1}(F_{n-m}C) + F_{n-m-1}(F_m C)$$

but since

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_n C , \begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m} C , \text{ and } \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_m C ,$$

we have derived

(D) $$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} .$$

Similarly, using identity (B), one can establish

(E) $$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} .$$

It is thus now easy to establish by mathematical induction that if the Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}$ are integers for an integer n $(m = 0, 1, \cdots, n)$, then they are integers for an integer $n + 1$ $(m = 0, 1, 2, \cdots, n + 1)$.

Recalling

$$L_m = F_{m+1} + F_{m-1} ,$$

then adding (D) and (E) yields

$$(3) \qquad 2\begin{bmatrix} n \\ m \end{bmatrix} = L_m \begin{bmatrix} n-1 \\ m \end{bmatrix} + L_{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} ,$$

where L_m is the m^{th} Lucas number, a result given in problem H-5, Fibonacci Quarterly Journal, Feb., 1963, page 47. From (3) it is harder to show that $\begin{bmatrix} n \\ m \end{bmatrix}$ is an integer.

With a slight change in notation, let us return to identities (A) and (B),

$$(A) \qquad F_{n'} = F_{m'+1}F_{n'-m'} + F_{m'}F_{n'-m'-1} ,$$

$$(B) \qquad F_{n'} = F_{m'}F_{n'-m'+1} + F_{m'-1}F_{n'-m'} .$$

For $k > 0$, let $n' = nk$ and $m' = mk$, then

$$(A') \qquad F_{nk} = F_{mk+1}F_{k(n-m)} + F_{mk}F_{k(n-m)-1} ,$$

$$(B') \qquad F_{nk} = F_{mk}F_{k(n-m)+1} + F_{mk-1}F_{k(n-m)} .$$

Let $u_n \equiv F_{nk}$. Then one can show, in a manner similar to above, using (A') and (B'), that if

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = \frac{u_n u_{n-1} \cdots u_2 u_1}{(u_m u_{m-1} \cdots u_2 u_1)(u_{n-m} u_{n-m-1} \cdots u_2 u_1)} , \quad 0 < m < n ,$$

and

$$\begin{bmatrix} n \\ n \end{bmatrix}_k = \begin{bmatrix} n \\ 0 \end{bmatrix}_k = 1 \;, \text{ then}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = F_{km+1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + F_{k(n-m)-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_k = F_{km-1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + F_{k(n-m)+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

or, adding these two,

$$2\begin{bmatrix} n \\ m \end{bmatrix}_k = L_{km}\begin{bmatrix} n-1 \\ m \end{bmatrix}_k + L_{k(n-m)}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_k \;,$$

a generalization of (3). We note here each u_n is divisible by F_k and we'd get the same generalized binomial coefficients from

$$u_n \equiv F_{nk} / F_k \;,$$

III. THE FIBONOMIAL TRIANGLE

Pascal's arithmetic triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$$\binom{n}{0} \quad \binom{n}{1} \cdots \binom{n}{m} \cdots \binom{n}{n-1} \quad \binom{n}{n}$$

has been the subject of many studies and has always generated interest. We note here to get the next line we merely use the recurrence relation

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} ,$$

Here we point out two interpretations, one of which shows a direction for Fibonacci generalization. The usual first meeting with Pascal's triangle lies in the binomial theorem expansion,

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j .$$

However, of much interest to us is the difference equation interpretation. The difference equation satisfied by n^0 is

$$(n + 1)^0 - n^0 = 0 ,$$

while the difference equation satisfied by n is

$$(n + 2) - 2(n + 1) + n = 0 .$$

For n^2 the difference equation is

$$(n+3)^2 - 3(n+2)^2 + 3(n+1)^2 - n^2 = 0 .$$

Certainly one notices the binomial coefficients with alternating signs appearing here. In fact,

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (n + m + 1 - j)^m = 0 .$$

It is this connection with the difference equations for the powers of the integers that leads us naturally to the Fibonomial triangle.

Similar to the difference equation coefficients array for the powers of the positive integers which results in Pascal's arithmetic triangle with alternating signs, there is the Fibonomial triangle made up of the Fibonomial coefficients, with doubly alternated signs. We first write down the Fibonomial triangle for the first six levels.

```
                  1
                1   1
              1   1   1
            1   2   2   1
          1   3   6   3   1
        1   5  15  15   5   1
      1   8  40  60  40   8   1
```

The top line is the 0^{th} row and the coefficients of the difference equation satisfied by F_n^k are the numbers in the $(k+1)^{st}$ row. Of course, we can get the next line of Fibonomial coefficients by using our recurrence relation (D),

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad 0 < m < n .$$

We now rewrite the Fibonomial triangle with appropriate signs so that the rows are properly signed to be the coefficients in the difference equations satisfied by F_n^k.

	1						
F_n^0 :	1	-1					
F_n^1 :	1	-1	-1				
F_n^2 :	1	-2	-2	+1			
F_n^3 :	1	-3	-6	+3	+1		
F_n^4 :	1	-5	-15	+15	+5	-1	
F_n^5 :	1	-8	-40	+60	+40	-8	-1

Thus, from the above we may write

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$F_{n+5}^4 - 5F_{n+4}^4 - 15F_{n+3}^4 + 15F_{n+2}^4 + 5F_{n+1}^4 - F_n^4 = 0 .$$

In Jarden [1] and Hoggatt and Hillman [2] is given the auxiliary polynomial for the difference equation satisfied by F_n^m,

$$\sum_{h=0}^{m+1} \begin{bmatrix} m+1 \\ h \end{bmatrix} (-1)^{h(h+1)/2} x^{m+1-h} ,$$

which shows that the sign pattern of doubly alternating signs persists. For an interesting related problem, see [5] and [6].

IV. THE GENERALIZED FIBONOMIAL TRIANGLE

If, instead of the Fibonacci Sequence, we consider the sequence

$$u_n \equiv F_{nk} \quad (k = 1, 2, 3, \cdots) \quad ,$$

there results another triangular array for each $k > 0$ which all have integer entries. We illustrate with F_{2n}. The recurrence relation is

$$\begin{bmatrix} n \\ m \end{bmatrix}_2 = F_{2m-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_2 + F_{2(m-n)+1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2$$

and

$$\begin{bmatrix} n \\ n \end{bmatrix}_2 = \begin{bmatrix} n \\ 0 \end{bmatrix}_2 = 1 \ .$$

The first few lines, with signs, are given below:

			1			
F_{2n}^0 :	1	-1				
F_{2n}^1 :	1	-3	+1			
F_{2n}^2 :	1	-8	+8	-1		
F_{2n}^3 :	1	-21	+56	-21	+1	
F_{2n}^4 :	1	-55	385	-385	+55	-1

We are saying that the difference equation satisfied by F_{2n}^4 is

$$F^4_{2n+10} - 55F^4_{2n+8} + 385F^4_{2n+6} - 385F^4_{2n+4} + 55F^4_{2n+2} - F^4_{2n} = 0 .$$

The algebraic signs of each triangle (singly alternating or doubly alternating) will be determined by the second row by the auxiliary polynomial of F_{kn} which is

$$x^2 - L_k x + (-1)^k .$$

For the general second-order recurrence relation

$$u_{n+2} = pu_{n+1} + qu_n , \quad q \neq 0 ,$$

the auxiliary polynomial is given in [2] to be

$$\sum_{h=0}^{m+1} (-1)^h \begin{bmatrix} m+1 \\ h \end{bmatrix} (-q)^{h(h-1)/2} x^{m+1-h} ,$$

where

$$\begin{bmatrix} m+1 \\ h \end{bmatrix}$$

is the generalized binomial coefficient which in our case becomes

$$\begin{bmatrix} m+1 \\ h \end{bmatrix}_k .$$

Thus for all generalized Fibonomial triangles the generalized Fibonomial coefficients with appropriate signs present arrays which are the coefficients of the difference equations satisfied by the powers, F_{kn}^m, of the Fibonacci sequence.

V. A GENERAL TECHNIQUE

Three simple pieces of information can be used to directly obtain the auxiliary polynomials for F_{kn}^m.

Lemma. If sequence u_n is such that

$$(E^2 + pE + q)u_n \equiv 0$$

and sequence v_n is such that

$$(E^2 + p'E + q')v_n \equiv 0 \, ,$$

where

$$x^2 + px + q = 0 \quad \text{and} \quad x^2 + p'x + q' = 0$$

have no common roots, then the sequence

$$w_n = Au_n + Bv_n$$

is such that

$$(E^2 + pE + q)(E^2 + p'E + q')w_n = 0 \, ,$$

for arbitrary constants A and B. See problem B-65, Fibonacci Quarterly Journal, April, 1965, page 153.

The auxiliary polynomial for F_{nk} is

$$x^2 - L_k x + (-1)^k \, .$$

The Binet Form for

$$F_m = (\alpha^m - \beta^m)/(\alpha - \beta)$$

and

$$L_m = \alpha^m + \beta^m ,$$

where

$$\alpha = (1 + \sqrt{5})/2 \quad \text{and} \quad \beta = (1 - \sqrt{5})/2 .$$

Suppose we wish to find the auxiliary polynomial associated with, say, F_{2n}^3.

$$F_{2n}^3 = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)^3 = \frac{\alpha^{6n} - 3\alpha^{4n}\beta^{2n} + 3\alpha^{2n}\beta^{4n} - \beta^{6n}}{5(\alpha - \beta)}$$

$$= \frac{1}{5}\left\{\frac{\alpha^{6n} - \beta^{6n}}{\alpha = \beta} - 3(\alpha\beta)^{2n}\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)\right\}$$

$$= \frac{1}{5}(F_{6n} - 3F_{2n}) .$$

Now, the auxiliary polynomial for $\frac{1}{5}F_{6n}$ is

$$x^2 - L_6 x + 1$$

and for $\frac{-3}{5}F_{2n}$ is

$$x^2 - L_2 x + 1 .$$

Thus the auxiliary polynomial associated with F_{2n}^3 is

$$(x^2 - 18x + 1)(x^2 - 3x + 1) = x^4 - 21x^3 + 56x^2 - 21x + 1 .$$

We illustrate the technique with F_n^5 .

$$F_n^5 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^5 = \frac{\alpha^{5n} - 5\alpha^{4n}\beta^n + 10\alpha^{3n}\beta^{2n} - 10\alpha^{2n}\beta^{3n} + 5\alpha^n\beta^{4n} - \beta^{5n}}{25(\alpha - \beta)}$$

$$= \frac{1}{25}\left(F_{5n} - 5(\alpha\beta)^n F_{3n} + 10(\alpha\beta)^{2n}F_n\right) .$$

The auxiliary polynomials are

$$\frac{1}{25}F_{5n} \quad : \qquad x^2 - L_5x - 1 = x^2 - 11x - 1$$

$$\frac{1}{5}(-1)^n F_{3n} : \qquad x^2 + L_3x - 1 = x^2 + 4x - 1$$

$$\frac{10}{25}F_n : \qquad x^2 - L_1x - 1 = x^2 - x - 1$$

so that the auxiliary polynomial for F_n^5 is

$$(x^2 - 11x - 1)(x^2 + 4x - 1)(x^2 - x - 1) = x^6 - 8x^5 - 40x^4 + 60x^3 + 40x^2 - 8x - 1$$

which the reader should check with the array in Section III with the Fibonomial Triangle.

This technique can thus be used to find the factored form or recurrence relationship for the auxiliary polynomials for any F_{nk}^m $(m = 0, 1, 2, \cdots)$. See [1] and [3] and particularly [4].

VI. THE GENERAL SECOND-ORDER RECURRENCE

Consider the sequence $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = pu_{n+1} + qu_n$, for $n \geq 0$. Define the generalized binomial coefficient

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{u_n u_{n-1} \cdots u_2 u_1}{(u_m u_{m-1} \cdots u_2 u_1)(u_{n-m} u_{n-m-1} \cdots u_2 u_1)} , \quad 1 \le m \le n - 1 ,$$

with

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1 , \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = 0 \text{ for } m > n \ge 0 \quad .$$

Starting with

$$R = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} ,$$

then

$$R^n = \begin{pmatrix} g_{n+1} & qg_n \\ g_n & qg_{n-1} \end{pmatrix}, \quad n \ge 1 ,$$

can be easily established by mathematical induction. Thus we can easily obtain, as in Section II, that

$$g_n = g_{m+1} g_{n-m} + q g_m g_{n-m-1}$$

$$g_n = g_m g_{n-m+1} + q g_{m-1} g_{n-m}$$

Thus, we can immediately write

$$\text{(F)} \qquad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = g_{m+1} \left\{ \begin{matrix} n - 1 \\ m \end{matrix} \right\} + q g_{n-m-1} \left\{ \begin{matrix} n - 1 \\ m - 1 \end{matrix} \right\}$$

and

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = q g_{m-1} \left\{ \begin{matrix} n - 1 \\ m \end{matrix} \right\} + g_{n-m+1} \left\{ \begin{matrix} n - 1 \\ m - 1 \end{matrix} \right\}$$

We can now examine some special cases. If $p = 2$ and $q = -1$, then $g_n = n$. The above identities become ordinary binomial coefficients,

$$\binom{n}{m} = (m + 1) \binom{n - 1}{m} - (n - m - 1) \binom{n - 1}{m - 1}$$

$$\binom{n}{m} = -(m - 1) \binom{n - 1}{m} + (n - m + 1) \binom{n - 1}{m - 1} .$$

and adding yields $$\binom{n}{m} = \binom{n - 1}{m} + \binom{n - 1}{m - 1} .$$

Thus we can conclude that the binomial coefficients are integers and that the product of any m consecutive positive integers is divisible by m-factorial. Since the Fibonomial coefficients are integers, then the product of any m consecutive Fibonacci numbers (with positive subscripts) is exactly divisible by the product of the first m Fibonacci numbers.

If, on the other hand, $p = x$ and $q = 1$, then $g_n(x) = f_n(x)$, the Fibonacci polynomials, and the rows of the generalized binomial coefficients array are indeed the coefficients, with doubly alternated signs, of the difference equations satisfied by the powers of the Fibonacci polynomials, which are $f_0(x) = 0$, $f_1(x) = 1$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, $n \geq 0$. The resulting generalized binomial coefficients are monic polynomials with integral coefficients.

VII. THE FIBONACCI POLYNOMIAL BINOMIAL COEFFICIENT TRIANGLE

The first few Fibonacci polynomials are

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2 + 1, \quad f_4(x) = x^3 + 2x, \quad f_5(x) = x^4 + 3x^2 + 1,$$

and the first few lines of the Fibonacci polynomial triangle are

$$\begin{array}{lccccccc}
 & & & & 1 & & & \\
f_n^0(x): & & & 1 & & -1 & & \\
f_n^1(x): & & 1 & & -x & & -1 & \\
f_n^2(x): & 1 & & -(x^2+1) & & -(x^2+1) & & +1 \\
f_n^3(x): & 1 & -(x^3+2x) & & -(x^2+1)(x^2+2) & & +(x^3+2x) & +1 \\
 & \left\{ {n \atop 0} \right\} & \left\{ {n \atop 1} \right\} & \cdots & \left\{ {n \atop m} \right\} & \cdots & \left\{ {n \atop n-1} \right\} & \left\{ {n \atop n} \right\},
\end{array}$$

where the double signs are to be attached to the $\left\{ {n \atop m} \right\}$ as required.

We are saying

$$f_{n+4}^3(x) - (x^3 + 2x)f_{n+3}^3(x) - (x^2+1)(x^2+2)f_{n+2}^3(x)$$

$$+ (x^3 + 2x)f_{n+1}^3(x) + f_n^3(x) \equiv 0 .$$

The next line can be obtained by using recurrence relation (F).

$$\text{(F)} \qquad \left\{ {n \atop m} \right\} = f_{m+1}(x) \left\{ {n-1 \atop m} \right\} + f_{n-m-1}(x) \left\{ {n-1 \atop m-1} \right\},$$

where

$$\left\{ {n \atop 0} \right\} = 1 = \left\{ {n \atop n} \right\} .$$

This triangular array collapses into the Fibonomial triangle when $x = 1$. From (F) it is easy to establish by induction that $\left\{ {n \atop m} \right\}$ are monic polynomials with integral coefficients. For every integral x we get an array of integers.

VIII. THE CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

The Chebyshev polynomials of the second kind are

$$u_0(x) = 1, \quad u_1(x) = 2x, \quad \text{and} \quad u_{n+2}(x) = 2x\, u_{n+1}(x) - u_n(x) .$$

If

$$g_n(x) = u_{n-1}(x),$$

then

$$g_0(x) = 0, \qquad g_1(x) = 1 ,$$

and we have the conditions for our Pascal triangle rows to have singly alternating signs to reflect the difference equations for the powers of $g_n(x)$. Since $g_n(x/2)$ also satisfies this, the Fibonacci polynomials and the Chebyshev polynomials yield all possible Pascal triangles with integral coefficients.

IX. THE FINAL DISCUSSION

In [1] and [2] it is given that the auxiliary polynomial associated with the general second-order recurrence

$$y_{n+2} = py_{n+1} + qy_n, \quad q \neq 0 ,$$

is

$$\sum_{h=0}^{m+1} (-1)^h \left\{ \begin{matrix} m+1 \\ h \end{matrix} \right\} (-q)^{h(h-1)/2} x^{m+1-h}$$

Thus, if the columns of Pascal's generalized binomial coefficient triangle is left justified with the first column on the left being the 0^{th} column then multiplying the h^{th} column by $q^{h(h-1)/2}$ yields a modified array whose coefficients along each row (with singly alternating signs if Chebyshev related or doubly alternating if Fibonacci related) are the coefficients of the difference equations satisfied by u_n^m.

REFERENCES

1. Dov Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem, Israel, 1958, pp. 42-45.
2. A. P. Hillman and V. E. Hoggatt, "The Characteristic Polynomial of the Generalized Shift Matrix," Fibonacci Quarterly 3:2, April, 1965, pp. 91-94.
3. Terrence A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly 2:2, April, 1964, pp. 93-103.
4. Stephen Jerbic, problem H-63, Fibonacci Quarterly 3:2, April, 1965, p. 116.
5. V. E. Hoggatt, Jr., problem H-76, Fibonacci Quarterly 3:4, Dec., 1965, p. 300.
6. N. A. Draim and Marjorie Bicknell, "Sums of n^{th} Powers of Roots of a Given Quadratic Equation," Fibonacci Quarterly 4:2, April, 1966, pp. 170-178.

* * * * *