A SHIFT FORMULA FOR RECURRENCE RELATIONS OF ORDER m

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It is well known that if F_i is the i^{th} Fibonacci number, then

$$F_{n+k+1} = F_{n+1}F_{k+1} + F_nF_k$$

for all integers n, k. A generalization of this identity to recurrence relations of any order m is given here.

Let m be a positive integer and let $p_1, p_2, \cdots, p_m$ $(p_m \neq 0)$ be m elements of a field F. Furthermore, let $\{y_i\}$ and $\{U_i\}$ be two sequences in F obeying the recurrence relation whose auxiliary polynomial is

$$P(x) = x^m - \sum_{j=0}^{m-1} p_{m-j}x^j ,$$

and let $\{U_i\}$ have the initial values

$$U_0 = U_1 = \cdots = U_{m-2} = 0$$

and

$$U_{m-1} = 1 .$$

Then,

$$y_{n+k} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} p_{m-i}U_{k+i-j-1}y_{n+j} \tag{1}$$

for all integers n and k.

The proof of (1) is by induction on k. Let n be fixed. For $0 \le k < m$ it is clear that

$$(2)\quad \sum_{i=0}^{j} p_{m-i}U_{k+i-j-1} = \begin{cases} 0 & \text{if } j < k \\ p_m U_{-1} = 1 & \text{if } j = k \\ \sum_{i=0}^{m-1} p_{m-i}U_{k+i-j-1} = U_{k+m-j-1} = 0 & \text{if } k < j < m . \end{cases}$$

From (2) it immediately follows that (1) holds for $k = 0, 1, \cdots, m-1$. From here, applications of the recurrence relation (corresponding to $P(x)$) for $\{y_i\}$ and $\{U_i\}$, in both the forward and backward directions, easily prove that if (1) holds for $k = h, h+1, \cdots, h+m-1$, then (1) holds for $k = h-1, h, \cdots, h+m$. By application of finite induction, it follows that (1) holds for all integers n, k.

Let $P(x) = (x - r_1)(x - r_2)\cdots(x - r_m)$ in an extension G of F and suppose that G is of characteristic zero. Further suppose that the r_j are pairwise distinct. Define D_k as the determinant produced by the process of substituting the vector $(r_1^k, r_2^k, \cdots, r_m^k)$ for the m^{th} row $(r_1^{m-1}, r_2^{m-1}, \cdots, r_m^{m-1})$ in the Vandermonde determinant of $r_1, r_2, \cdots, r_m$. It is proven in [1] that for every integer k,

$$(3)\qquad U_k = \frac{D_k}{D_{m-1}} .$$

The case for repetitions among the r_j is handled in the following way: Start with the form for U_k in (3) and, pretending that the r_j are real, apply L'Hospital's Rule successively as $r_I \to r_J$ for all repetitions $r_I = r_J$ among the r_j.

A combination of (1) and (3) now comes with ease. Still taking the r_j to be pairwise distinct, define E_k as the determinant produced by the process of replacing the element r_h^k of the m^{th} row of D_k by

$$\sum_{j=0}^{m-1}\sum_{i=0}^{j} p_m y_j r_h^{k+i-j-1} ,$$

and this for $h = 1, 2, \cdots, m$. Then combination of (1) with (3) yields: For every integer k,

$$y_k = \frac{E_k}{D_{m-1}} \qquad (4)$$

The case for repeated roots is handled as with (3). In [2] identities akin to (4) are developed.

REFERENCES

1. Arkin, Joseph, "Recurring Series," to appear in the Fibonacci Quarterly.
2. Styles, C. C., "On Evaluating Certain Coefficients," The Fibonacci Quarterly, Vol. 4, No. 2, April, 1966.

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