A PRIMER FOR THE FIBONACCI NUMBERS: PART VI

V. E. HOGGATT, JR., AND D. A. LIND
San Jose State College, San Jose, California, and University of Virginia, Charlottesville, Va.

1. INTRODUCTION

We shall devote this part of the primer to the topic of generating functions. These play an important role both in the general theory of recurring sequences and in combinatorial analysis. They provide a tool with which every Fibonacci enthusiast should be familiar.

2. GENERAL THEORY OF GENERATING FUNCTIONS

Let $a_0, a_1, a_2, \cdots$ be a sequence of real numbers. The ordinary generating function of the sequence $\{a_n\}$ is the series

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n .$$

Another type of generating function of great use in combinatorial problems involving permutations is the exponential generating function of $\{a_n\}$, namely

$$E(x) = a_0 + a_1x/1! + a_2x^2/2! + \cdots = \sum_{n=0}^{\infty} a_n x^n/n! .$$

For some examples of the two types of generating functions, first let $a_n = a^n$. The ordinary generating function of $\{a_n\}$ is then the geometric series

$$A(x) = \frac{1}{1 - ax} = \sum_{n=0}^{\infty} a^n x^n , \tag{2.1}$$

while the exponential generating function is

$$E(x) = e^{ax} = \sum_{n=0}^{\infty} a^n x^n / n! \quad .$$

Similarly, if $a_n = na^n$, then

(2.2)

$$A(x) = \frac{ax}{(1 - ax)^2} = \sum_{n=0}^{\infty} na^n x^n \quad ,$$

$$E(x) = axe^{ax} = \sum_{n=0}^{\infty} na^n x^n / n! \quad ,$$

each of these being obtained from the preceding one of the same type by differentiation and multiplication by x. A good exercise for the reader to check his understanding is to verify that if $a_n = n^2$, then

$$A(x) = \frac{x(x + 1)}{(1 - x)^3} = \sum_{n=0}^{\infty} n^2 x^n \quad ,$$

$$E(x) = x(x + 1)e^x = \sum_{n=0}^{\infty} n^2 x^n / n! \quad .$$

(Hint: Differentiate the previous results again.)

For the rest of the time, however, we will deal exclusively with ordinary generating functions.

We adopt the point of view here that x is an indeterminant, a means of distinguishing the elements of the sequence through its powers. Used in this context, the generating function becomes a tool in an algebra of these sequences (see [3]). Then formal operations, such as addition, multiplication, differentiation with respect to x, and so forth, and equating equations of like powers of x after these operations merely express relations in this algebra, so that convergence of the series is irrelevant.

The basic rules of manipulation in this algebra are analogous to those for handling polynomials. If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences with (ordinary) generating functions $A(x)$, $B(x)$, $C(x)$ respectively, then $A(x) + B(x) = C(x)$ if and only if $a_n + b_n = c_n$, and $A(x)B(x) = C(x)$ if and only if

$$c_n = a_nb_0 + a_{n-1}b_1 + \cdots + a_1b_{n-1} + a_0b_n .$$

Both results are obtained by expanding the indicated sum or product of generating functions and comparing coefficients of like powers of x. The product here is called the Cauchy product of the sequences $\{a_n\}$ and $\{b_n\}$, and the sequence $\{c_n\}$ is called the convolution of the two sequences $\{a_n\}$ and $\{b_n\}$.

To give an example of the usefulness and convenience of generating functions, we shall derive a well-known but nontrivial binomial identity. First note that for a fixed real number k the generating function for the sequence

$$a_n = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

is

$$A_k(x) = (1 + x)^k$$

by the binomial theorem. If k is a nonnegative integer, the generating function is finite since

$$\binom{k}{n} = 0 \quad \text{if} \quad n > k \geq 0 \quad \text{or} \quad n < 0 \tag{2.3}$$

by definition. Then

$$A_k(x) = (1 + x)^k = (1 + x)^{k-m}(1 + x)^m = A_{k-m}(x)A_m(x) .$$

Using the product rule gives

$$\sum_{n=0}^{k}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \left(\sum_{n=0}^{\infty}\binom{k-m}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{m}{n}x^n\right)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j}\right]x^n ,$$

so that equating coefficients of x^n shows

$$\binom{k}{n} = \sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j} .$$

This can be found in Chapter 1 of [8].

If the generating function for $\{a_n\}$ is known, it is sometimes desirable to convert it to the generating function for $\{a_{n+k}\}$ as follows. If

$$A(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

then

$$\frac{A(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1}x^n .$$

This can be repeated as often as needed to obtain the generating function for $\{a_{n+k}\}$.

Generating functions are a powerful tool in the theory of linear recurring sequences and the solution of linear difference equations. As an example, we shall solve completely a second-order linear difference equation using the technique of generating functions. Let $\{c_n\}$ be a sequence of real numbers which obey

$$c_{n+2} - pc_{n+1} + qc_n = 0, \quad n \geq 0,$$

where c_0 and c_1 are arbitrary. Then by using the Cauchy product we find

$$(1 - px + qx^2)\sum_{n=0}^{\infty} c_n x^n = c_0 + (c_1 - pc_0)x + 0 \cdot x^2 + \cdots$$

$$= c_0 + (c_1 - pc_0)x = r(x) \quad ,$$

so that

(2.4) $$\sum_{n=0}^{\infty} c_n x^n = \frac{r(x)}{1 - px + qx^2} \quad .$$

Suppose a and b are the roots of the auxiliary polynomial $x^2 - px + q$, so the denominator of the generating function factors as $(1 - ax)(1 - bx)$. We divide the treatment into two cases, namely, $a \neq b$ and $a = b$.

If a and b are distinct (i. e., $p^2 - 4q \neq 0$), we may split the generating function into partial functions, giving

(2.5) $$\frac{r(x)}{1 - px + qx^2} = \frac{r(x)}{(1 - ax)(1 - bx)} = \frac{A}{1 - ax} + \frac{B}{1 - bx}$$

for some constants A and B. Then using (2.1) we find

$$\sum_{n=0}^{\infty} c_n x^n = A\sum_{n=0}^{\infty} a^n x^n + B\sum_{n=0}^{\infty} b^n x^n = \sum_{n=0}^{\infty} (Aa^n + Bb^n)x^n \quad ,$$

so that an explicit formula for c_n is

(2.6) $$c_n = Aa^n + Bb^n \; .$$

Here A and B can be determined from the initial conditions resulting from assigning values to c_0 and c_1.

On the other hand, if the roots are equal (i.e., $p^2 - 4q = 0$), the situation is somewhat different because the partial fraction expansion (2.5) is not valid. Letting $r(x) = r + sx$, we may use (2.2), however, to find

$$\sum_{n=0}^{\infty} c_n x^n = \frac{r + sx}{(1 - ax)^2} = (r + sx) \sum_{n=0}^{\infty} (n + 1)a^n x^n$$

$$= \sum_{n=0}^{\infty} (r(n + 1)a^n + sna^{n-1})x^n = \sum_{n=0}^{\infty} ((r + s/a)n + r)a^n x^n ,$$

showing that

$$c_n = (An + B)a^n ,$$

where

$$A = r + s/a, \qquad B = r$$

are constants which again can be determined from the initial values c_0 and c_1.

This technique can be easily extended to recurring sequences of higher order. For further developments, the reader is referred to Jeske [6], where a generalized version of the above is derived in another way. For a discussion of the general theory of generating functions, see Chapter 2 of [8] and Chapter 3 of [2].

3. APPLICATIONS TO FIBONACCI NUMBERS

The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} - F_{n+1} - F_n = 0$, $n \ge 0$. Using the general solution of the second-order difference equation given above, where $p = 1$, $q = -1$, $r(x) = x$, we find that the generating function for the Fibonacci numbers is

$$F(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad . \tag{3.1}$$

The reader should actually divide out the middle part of (3.1) by long division to see that Fibonacci numbers really do appear as coefficients.

Since the roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ of the auxiliary polynomial $x^2 - x - 1$ are distinct, we see from (2.6) that

$$F_n = A\alpha^n + B\beta^n \quad . \tag{3.2}$$

Putting $n = 0, 1$ and solving the resulting system of equations shows that

$$A = 1/\sqrt{5} = 1/(\alpha - \beta), \quad B = -1/\sqrt{5} \quad ,$$

establishing the familiar Binet form,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad . \tag{3.3}$$

We shall now turn around and use this form to derive the original generating function (3.1) by using a technique first exploited by H. W. Gould [5]. Suppose that some sequence $\{a_n\}$ has the generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad .$$

Then

$$\frac{A(\alpha x) - A(\beta x)}{\alpha - \beta} = \sum_{n=0}^{\infty} a_n \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) x^n = \sum_{n=0}^{\infty} a_n F_n x^n \quad . \tag{3.4}$$

In particular, if $a_n = 1$, then $A(x) = 1/(1 - x)$, so that

$$F(x) = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right) = \frac{x}{1 - x - x^2} .$$

Next we use (3.1) to prove that the Fibonacci numbers are the sums of terms along the rising diagonals of Pascal's Triangle. We write

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} = \frac{x}{1 - (x + x^2)} = x\sum_{n=0}^{\infty} x^n (1 + x)^n$$

$$= \sum_{n=0}^{\infty} x^{n+1} \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n+k+1}$$

$$= \sum_{m=1}^{\infty} \sum_{j=0}^{[(m-1)/2]} \binom{m - j - 1}{j} x^m ,$$

where $[m]$ denotes the greatest integer contained in m. The inner sum is the sum of coefficients of x^m in the preceding sum, and the upper limit of summation is determined by the inequality $m - j - 1 < j$, recalling (2.3). The reader is urged to carry through the details of this typical generating function calculation. Equating coefficients x^n shows that

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n - j - 1}{j} \tag{3.5}$$

linking the Fibonacci numbers to the binomial coefficients.

It follows from (3.1) upon division by x that

$$G(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n . \tag{3.6}$$

Differentiating this yields

$$G'(x) = \frac{2x + 1}{(1 - x - x^2)^2} = \left(\frac{1}{1 - x - x^2}\right)\left(\frac{1+2x}{1 - x - x^2}\right) = \sum_{n=0}^{\infty}(n+1)F_{n+2}x^n.$$

Now

$$\frac{1 + 2x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_{n+1}\, x^n \quad ,$$

where the L_n are the Lucas numbers defined by $L_1 = 1$,

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0.$$

Hence

$$G'(x) = \left(\sum_{n=0}^{\infty} F_{n+1}\, x^n\right)\left(\sum_{n=0}^{\infty} L_{n+1} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{n-k+1} L_{k+1}\right) x^n ,$$

so that

$$\sum_{k=0}^{n} F_{n-k+1} L_{k+1} = (n + 1)F_{n+2} \quad ,$$

a convolution of the Fibonacci and Lucas sequences.

We leave it to the reader to verify that

$$\frac{x}{(1 - x)(1 - x - x^2)} = \frac{x}{1 - 2x + x^3} = \sum_{n=0}^{\infty}(F_{n+2} - 1)x^n \quad .$$

Also

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{1}{1-x}\quad\frac{x}{1-x-x^2} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} F_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} F_j\right) x^n .$$

Equating coefficients shows

$$\sum_{j=0}^{n} F_j = F_{n+2} - 1 ,$$

which is really the convolution of the Fibonacci sequence with the constant sequence $\{1, 1, 1, \cdots\}$.

Consider the sequence $\{F_{kn}\}_{n=0}^{\infty}$, where $k \neq 0$ is an arbitrary but fixed integer. Since

$$F_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta}$$

we have

$$\sum_{n=0}^{\infty} F_{kn}x^n = \frac{1}{\alpha-\beta}\left(\sum_{n=0}^{\infty} \alpha^{kn}x^n - \sum_{n=0}^{\infty} \beta^{kn}x^n\right)$$

$$(3.7)\qquad = \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha^k x} - \frac{1}{1-\beta^k x}\right) = \frac{1}{\alpha-\beta}\left(\frac{(\alpha^k - \beta^k)\,x}{1-(\alpha^k+\beta^k)x + (\alpha^k\beta^k)x^2}\right)$$

$$= \frac{F_k\, x}{1 - L_k x + (-1)^k x^2} ,$$

where we have used $\alpha\beta = -1$ and the Binet form $L_n = \alpha^n + \beta^n$ for the Lucas numbers. Incidentally, since here the integer in the numerator must divide all coefficients in the expansion, we have a quick proof that F_k divides F_{nk} for all n. A generalization of (3.7) is given in equation (4.18) of Section 4.

We turn to generating functions for powers of the Fibonacci numbers. First we expand

$$F_n^2 = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 = \frac{1}{(\alpha - \beta)^2}\left(\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}\right) .$$

Then

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{1}{(\alpha - \beta)^2}\left(\sum_{n=0}^{\infty} \alpha^{2n}x^n - 2\sum_{n=0}^{\infty}(\alpha\beta)^n x^n + \sum_{n=0}^{\infty}\beta^{2n}x^n\right)$$

$$= \frac{1}{(\alpha - \beta)^2}\left(\frac{1}{1 - \alpha^2 x} - \frac{2}{1 - \alpha\beta x} + \frac{1}{1 - \beta^2 x}\right)$$

$$= \frac{x - x^2}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \beta^2 x)} = \frac{x - x^2}{1 - 2x - 2x^2 + x^3}$$

This also shows that $\left\{F_n^2\right\}$ obeys

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 .$$

We remark that Gould's technique (3.3) may be applied to $F(x)$, and leads to exactly the same result.

In general, to find the generating function for the p^{th} power of the Fibonacci numbers, first expand F_n^p by the binomial theorem. This gives F_n^p as a linear combination of α^{np}, $\alpha^{n(p-1)}\beta^n, \cdots,$ $\alpha^n\beta^{n(p-1)}$, β^{np} so that as above the generating function will have the denominator

$$(1 - \alpha^p x)(1 - \alpha^{p-1}\beta x) \cdots (1 - \alpha\beta^{p-1}x)(1 - \beta^p x) .$$

Fortunately, this product can be expressed in a better way. Define the Fibonomial coefficients $\left[\begin{matrix}k\\r\end{matrix}\right]$ by

$$\begin{bmatrix}k\\r\end{bmatrix} = \frac{F_kF_{k-1}\cdots F_{k-r+1}}{F_1F_2\cdots F_r} \quad (r > 0); \quad \begin{bmatrix}k\\0\end{bmatrix} = 1 .$$

Then it has been shown [7] that

$$Q_p(x) = \prod_{j=0}^{p} (1 - \alpha^{p-j}\beta^j x) = \sum_{j=0}^{p+1} (-1)^{j(j+1)/2} \begin{bmatrix}p+1\\j\end{bmatrix} x^j .$$

For example,

$$\begin{aligned} Q_1(x) &= 1 - x - x^2 \\ Q_2(x) &= 1 - 2x - 2x^2 + x^3 \\ Q_3(x) &= 1 - 3x - 6x^2 + 3x^3 + x^4 \\ Q_4(x) &= 1 - 5x - 15x + 15x^3 + 5x^4 - x^5 . \end{aligned}$$

Since any sequence obeying the Fibonacci recurrence relation can be written in the form $A\alpha^n + B\beta^n$, $Q_p(x)$ is the denominator of the generating function of the p^{th} power of any such sequence. The numerators of the generating functions can be found by simply multiplying through $Q_p(x)$. For example, to find the generating function of $\{F_{n+2}^2\}$, we have

$$\sum_{n=0}^{\infty} F_{n+2}^2 x^n = \frac{r(x)}{1 - 2x - 2x^2 + x^3} .$$

Then $r(x)$ can be found by multiplying $Q_2(x)$, giving

$$\begin{aligned} r(x) &= (1 - 2x - 2x^2 + x^3)(1 + 4x + 9x^2 + 25x^4 + \cdots) \\ &= 1 + 2x - x^2 + 0 \cdot x^3 + \cdots = 1 + 2x - x^2 . \end{aligned}$$

This is (4.7) of Section 4. However, for fixed p, once we have obtained the generating functions for $\{F_n^p\}$, $\{F_{n+1}^p\}$, $\cdots$, $\{F_{n+p}^p\}$, the one for $\{F_{n+k}^p\}$ follows directly from the identity of Hoggatt and Lind [4]

$$F_{n+k}^p = \sum_{j=0}^{p} (-1)^{(p-j)(p-j+3)/2} \left[{k \atop p}\right] \left[{p \atop j}\right] \left(\frac{F_{k-p}}{F_{k-j}}\right) F_{n+j}^p , \tag{3.5}$$

where we use the convention $F_0/F_0 = 1$. For example, for $p = 1$ this gives

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n .$$

Using the generating function for $\{F_{n+1}\}$ in (3.4) and $\{F_n\}$ in (3.1), we get

$$\sum_{n=0}^{\infty} F_{n+k}\, x^n = F_k \sum_{n=0}^{\infty} F_{n+1}\, x^n + F_{k-1} \sum_{n=0}^{\infty} F_n\, x^n$$

$$= \frac{F_k + F_{k-1}x}{1 - x - x^2} .$$

In fact, one of the main purposes for deriving (3.5) was to express the generating function of $\{F_{n+k}^p\}$ as a linear combination of those of $\{F_n^p\}, \cdots, \{F_{n+p}^p\}$.

Alternatively, to obtain the generating function of $\{F_{n+k}^p\}$ from that of $\{F_n^p\}$, we could apply k times in succession the technique mentioned in Section 2 of finding the generating function of $\{a_{n+1}\}$ from that of $\{a_n\}$.

The generating function of powers of the Fibonacci numbers have been investigated by several authors (see [3], [5], and [7]).

4. SOME STANDARD GENERATING FUNCTIONS

We list here for reference some of the generating functions we have already derived along with others which can be established in the same way.

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n\, x^n \tag{4.1}$$

(4.2) $$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n$$

(4.3) $$\frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n$$

(4.4) $$\frac{1 + 2x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_{n+1} x^n$$

(4.5) $$\frac{x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n^2 x^n$$

(4.6) $$\frac{1 - x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+1}^2 x^n$$

(4.7) $$\frac{1 + 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+2}^2 x^n$$

(4.8) $$\frac{x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n F_{n+1} x^n$$

(4.9) $$\frac{4 - 7x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_n^2 x^n$$

(4.10) $$\frac{1 + 7x - 4x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+1}^2 x^n$$

(4.11) $$\frac{9 - 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+2}^2 x^n$$

(4.12) $$\frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n^3 x^n$$

(4.13) $$\frac{1 - 2x - x^2}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+1}^3 x^n$$

(4.14) $$\frac{1 + 5x - 3x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+2}^3 x^n$$

(4.15) $$\frac{8 + 3x - 4x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+3}^3 x^n$$

(4.16) $$\frac{2x}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n F_{n+1} F_{n+2} x^n$$

(4.17) $$\frac{F_k x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn} x^n$$

(4.18) $$\frac{F_r + (-1)^r F_{k-r} x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+r} x^n$$

Many thanks to Kathleen Weland and Allan Scott.

REFERENCES

1. S. L. Basin and V. E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence, Part I," The Fibonacci Quarterly, 1 (1963), No. 1, pp. 65-72; Part II, 1 (1963), No. 2, pp. 61-68.

2. Edwin F. Beckenbach, editor, Applied Combinatorial Mathematics, Wiley, 1964.

3. L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," Duke Math. Journal, 29 (1962), pp. 521-538.

4. V. E. Hoggatt, Jr., and D. A. Lind, "A Power Identity for Second-Order Recurrent Sequences," The Fibonacci Quarterly, 4 (1966), pp. 274-282.

5. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," The Fibonacci Quarterly, 1 (1963), No. 2, pp. 1-16.

6. James A. Jeske, "Linear Recurrence Relations — Part I," The Fibonacci Quarterly, 1 (1963), No. 2, pp. 69-74.

7. John Riordan, "Generating Functions for Powers of Fibonacci Numbers," Duke Math. Journal, 29 (1962), pp. 5-12.

8. John Riordan, An Introduction to Combinatorial Analysis, Wiley, 1960.

* * * * *

The Fibonacci Bibliographical Research Center desires that any reader finding a Fibonacci reference send a card giving the reference and a brief description of the contents. Please forward all such information to:

Fibonacci Bibliographical Research Center,
Mathematics Department,
San Jose State College,
San Jose, California

* * * * *