GENERALIZED FIBONACCI SUMMATIONS

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INTRODUCTION

The operator Δ_r is defined [1] by:

$$\Delta_r f(r, a, b \cdots) = f(r, a, b \cdots) - f(r-1, a, b \cdots)$$

and its inverse Σ_r is defined by:

$$\Delta_r \Sigma_r f(r, a, b \cdots) = f(r, a, b \cdots)$$

In this article we will make use of these two operators, which are analogous to the differential and integral operators, to establish several summations involving generalized Fibonacci numbers.

First some elementary properties of Δ_r and Σ_r will be needed. In deriving these and in subsequent work the subscripts to the operators may be omitted if this causes no confusion.

PROPERTIES OF Δ_r AND Σ_r

1. $$\begin{aligned} \Delta(f(r) + g(r)) &= (f(r) + g(r)) - (f(r-1) + g(r-1)) \\ &= (f(r) - f(r-1)) + (g(r) - g(r-1)) \end{aligned}$$

(0.1) $$\Delta(f(r) + g(r)) = \Delta f(r) + \Delta g(r)$$

2. $$\begin{aligned} \Delta(f(r) \cdot g(r)) &= f(r) \cdot g(r) - f(r-1) \cdot g(r-1) \\ &= f(r) \cdot (g(r) - g(r-1)) + g(r-1) \cdot (f(r) - f(r-1)) \end{aligned}$$

(0.2) $$(f(r) \cdot g(r)) = f(r)\Delta g(r) + g(r-1)\Delta f(r)$$

If $g(r)$ is a constant then $\Delta_r g(r) = 0$ and putting $g(r) = C$ in (0.2) we have:

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(0.3) $$\Delta_r Cf(r) = C\Delta_r f(r) \text{ if } \Delta_r C = 0$$

This covers not only the case when C is a constant but also when it is any function independent of r.

(0.4) $$\Delta_n f(n + p) = (\Delta_r f(r))_{r=n+p}$$

This follows immediately from the definition of Δ_r sinch both left- and right-hand members simplify to $f(n + p) - f(n + p - 1)$.

4. Next some properties of Σ_r. Suppose: $\Sigma f(r) = g(r)$. Then from the definitions of Δ and Σ:

$$g(r) - g(r - 1) = f(r)$$

Summing these equalities with r taking values from 1 to n

$$g(n) - g(0) = \sum_{r=1}^{n} f(r)$$

i. e.,

(0.5) $$\Sigma f(n) = \sum_{r=1}^{n} f(r) + C ,$$

where $\Delta_n C = 0$ but otherwise C is arbitrary. The connection between the and the summation of $f(n)$ is equivalent to that between indefinite and definite integrals. In particular:

(0.6) $$\sum_{r=1}^{n} f(r) = \Sigma f(n) - (\Sigma f(n))_{n=0}$$

5. From (0.5)

$$\Sigma_n f(n + s) = \sum_{r=1}^{n} f(r + s) + C$$

$$= \sum_{r=1}^{n+s} f(r) + C - \sum_{r=1}^{s} f(r) = \sum_{r=1}^{n+s} f(r) + C'$$

If we ignore the constants:

(0.7) $$\Sigma_n f(n + s) = (\Sigma_r f(r))_{r=n+s}$$

6. In the definition of Σ put $\Delta f(r)$ in place of $f(r)$

$$\Delta(\Sigma\Delta f(r)) = \Delta(f(r))$$

i.e.,

$$\Sigma\Delta f(r) = f(r) + C$$

If we now ignore the constants

(0.8) $$\Sigma\Delta f(r) = f(r)$$

7. In (0.1) replace $f(r)$ by $\Sigma f(r)$ and $g(r)$ by $\Sigma g(r)$

$$\Delta(\Sigma f(r) + \Sigma g(r)) = \Delta\Sigma f(r) + \Delta\Sigma g(r)$$

$$\Sigma\Delta(\Sigma f(r) + \Sigma g(r)) = \Sigma(\Delta\Sigma f(r) + \Delta\Sigma g(r))$$

i.e.,

(0.9) $$\Sigma(f(r) + g(r)) = \Sigma f(r) + \Sigma g(r)$$

8. From (0.2) replace $g(r)$ by $h(r)$ and rearranging:

$$f(r)\Delta h(r) = \Delta(f(r) \cdot h(r)) - h(r-1)\Delta f(r)$$

Let $h(r) = \Sigma g(r)$

$$f(r) \cdot g(r) = \Delta(f(r) \cdot \Sigma g(r)) - \Sigma g(r-1) \cdot \Delta f(r)$$

Thus:

$$\Sigma(f(r) \cdot g(r)) = f(r)\Sigma g(r) - \Sigma(\Sigma g(r-1) \cdot \Delta f(r)) \tag{0.10}$$

This last result, analogous to integration by parts, will be made use of in deriving most of the results which follow.

If $f(r) = C$ where $\Delta_r C = 0$ we can write (0.10) as:

$$\Sigma C g(r) = C\Sigma g(r) \tag{0.11}$$

THE SUMMATIONS

The generalized Fibonacci numbers may be defined by:

$$H_n = H_{n-1} + H_{n-2} \tag{1.1}$$

for all integers n. If $H_0 = 0$ and $H_1 = 1$ we get the Fibonacci sequence which is denoted (F_n).

Two facts about the generalized sequence will be needed. They are:

$$H_{n-1}H_{n+1} - H_n^2 = D(-1)^n \quad \text{where } D = H_{-1}H_1 - H_0^2 \quad [2] \tag{1.2}$$

and

$$H_{n+r} = F_{r-1}H_n + F_rH_{n+1} \tag{1.3}$$

1. First a very simple (but useful) summation.

$$\Delta H_n = H_n - H_{n-1} = H_{n-2}$$

Thus:

$$\Sigma H_n = H_{n+2} \tag{1.4}$$

2. $\Sigma a^n H_{n+s}$

Note that

$$\Delta a^n = a^n - a^{n-1} = a^{n-1}(a - 1)$$

$$\begin{aligned}\Sigma a^n H_{n+s} &= a^n H_{n+s+2} - \Sigma a^{n-1}(a - 1)H_{n+s+1} \\ &= a^n H_{n+s+2} - \frac{a - 1}{a^2} \Sigma a^{n+1} H_{n+s+1}\end{aligned}$$

Now using:

$$\Sigma a^{n+1} H_{n+s+1} = \Sigma a^n H_{n+s} + a^{n+1} H_{n+s+1}$$

$$\frac{a^2 + a - 1}{a^2} \Sigma a^n H_{n+s} = a^n H_{n+s+2} - a^{n-1}(a - 1)H_{n+s+1}$$

multiplying by a^2

$$(a^2 + a - 1)\Sigma a^n H_{n+s} = a^{n+2} H_{n+s} + a^{n+1} H_{n+s+1}$$

If $a^2 + a - 1 \neq 0$ i.e., $a \neq (-1 \pm \sqrt{5})/2$

$$\Sigma a^n H_{n+s} = \frac{a}{a^2 + a - 1}(a^{n+1} H_{n+s} + a^n H_{n+s+1}) \tag{1.5}$$

3. $\Sigma n^k H_{n+s}$

Before attempting this summation we will find the particular sums when $k = 0, 1, 2.$

k=0: this comes straight from (1.4)

$$\Sigma H_{n+s} = H_{n+s+2} \tag{1.6}$$

k=1:

$$\Sigma nH_{n+s} = nH_{n+s+2} - \Sigma H_{n+s+1}$$

$$= nH_{n+s+2} - H_{n+s+3} \tag{1.7}$$

k=2:

$$\Sigma n^2 H_{n+s} = n^2 H_{n+s+2} - \Sigma(2n - 1)H_{n+s+1}$$

$$= n^2 H_{n+s+2} - 2nH_{n+s+3} + 2H_{n+s+4} + H_{n+s+3}$$

$$= (n^2 + 2)H_{n+s+2} + (3 - 2n)\, H_{n+s+3} \tag{1.8}$$

Results (1.6), (1.7) and (1.8) suggest that there is a general form:

$$\Sigma n^k H_{n+s} = A_k H_{n+s+2} + B_k H_{n+s+3} \tag{1.9}$$

where A_k, B_k are polynomials in n [3].
To determine the form of these polynomials consider:

$$\Sigma n^k H_{n+s} = n^k H_{n+s+2} - \Sigma(\Delta n^k)H_{n+s+1} \tag{1.10}$$

Now

$$\Delta n^k = n^k - \sum_{r=0}^{k} (-1)^r \binom{k}{r} n^{k-r} = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} n^{k-r}$$

(1.10) now becomes:

$$\Sigma n^k H_{n+s} = n^k H_{n+s+2} + \left(\sum_{r=1}^{k} (-1)^r \binom{k}{r} n^{k-r} \right) H_{n+s+1}$$

$$= n^k H_{n+s+2} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} H_{n+s+3} + B_{k-r} H_{n+s+4})$$

$$= \left(n^k + \sum_{r=1}^{k} (-1)^r \binom{k}{r} B_{k-r} \right) H_{n+s+2}$$

$$+ \left(\sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} + B_{k-r}) \right) H_{n+s+3}$$

Compare this with (1.9) and we have:

$$(1.11) \qquad \begin{aligned} A_k &= n^k + \sum_{r=1}^{k} (-1)^r \binom{k}{r} B_{k-r} \\ B_k &= \sum_{r=1}^{k} (-1)^r \binom{k}{r} (A_{k-r} + B_{k-r}) \end{aligned}$$

(1.11) and $A_0 = 1$; $B_0 = 0$ give us a way to find A_k, B_k for any non-negative integer k. Using (1.9) we then have the required sum. This is not a very convenient formula to deal with as the values of A_k, B_k given at the end of this article clearly show.

4. $\Sigma H_n H_{n+s}$

This form is chosen rather than one with $n + u$ and $n + v$ as subscripts because we can obtain this sum by putting $n + u$ in place of n and letting $s = v - u$.

Consider:

$$\Delta H_n H_{n+s} = H_n H_{n+s-2} + H_{n+s-1} H_{n-2}$$

(a) put $s = 1$

$$\Delta H_n H_{n+1} = H_n^2 \quad \text{i.e.,} \quad \Sigma H_n^2 = H_n H_{n+1}$$

(b) put $s = 0$

$$\Delta H_n^2 = H_{n-2}H_{n+1} \text{ i.e., } \Sigma H_n H_{n+3} = H_{n+2}^2$$

Combining these last two together

$$\Sigma H_n(AH_n + BH_{n+3}) = AH_nH_{n+1} + BH_{n+2}^2 \tag{1.12}$$

Now

$$AH_n + BH_{n+3} = (A + B)H_n + 2BH_{n+1}$$

so recalling (1.3) we can make (1.12) the required sum if

$$A + B = F_{s-1} \text{ and } 2B = F_s .$$

Let

$$B = \tfrac{1}{2}F_s \quad \text{and} \quad A = F_{s-1} - \tfrac{1}{2}F_s = \tfrac{1}{2}F_{s-3} :$$

(1.12) becomes:

$$\Sigma H_nH_{n+s} = \tfrac{1}{2}(F_{s-3}H_nH_{n+1} + F_sH_{n+2}^2) \tag{1.13}$$

5. $\Sigma H_nH_{n+r}H_{n+s}$

Let

$$h(n) = H_{n-1}H_{n+1} - H_n^2 = D(-1)^n$$

see (1.2)

$$H_{n-1}H_nH_{n+1} - H_n^3 = h(n)H_n$$

Now

$$\Sigma h(n)H_n = D\Sigma(-1)^nH_n = D(-1)^nH_{n-1}$$

from (1.5)

Thus:

$$\Sigma H_{n-1}H_nH_{n+1} - \Sigma H_n^3 = D(-1)^n H_{n-1} \tag{1.14}$$

We can sum H_n^3 by parts:

$$\Sigma H_n^3 = H_n \cdot H_nH_{n+1} - \Sigma H_{n-2}H_{n-1}H_n$$

Rearranging:

$$\Sigma H_{n-1}H_nH_{n+1} + \Sigma H_n^3 = H_n^2 H_{n+1} + H_{n-1}H_nH_{n+1} = H_nH_{n+1}^2 \tag{1.15}$$

From (1.14) and (1.15) we have:

$$\Sigma H_{n-1}H_nH_{n+1} = \tfrac{1}{2}(H_nH_{n+1}^2 + D(-1)^n H_{n-1}) \tag{1.16}$$

and:

$$\Sigma H_n^3 = \tfrac{1}{2}(H_nH_{n+1}^2 - D(-1)^n H_{n-1})$$

We now have two particular cases of the summation required. If we had

$$\Sigma H_n^2 H_{n+1}$$

as well as

$$\Sigma H_n^3$$

then by using the method of Section 4, we could generate $\Sigma H_n^2 H_{n+r}$

$$\Sigma H_n^2 H_{n+1} = H_{n+1} \cdot H_nH_{n+1} - \Sigma H_{n-1}H_n \cdot H_{n-1}$$

$$= H_nH_{n+1}^2 - \Sigma H_n^2 H_{n+1} + H_n^2 H_{n+1}$$

Thus:

$$\Sigma H_n^2 H_{n+1} = \tfrac{1}{2} H_n H_{n+1} H_{n+2} \tag{1.17}$$

Combining this with H_n^3 as promised:

$$\Sigma H_n^2 H_{n+r} = \tfrac{1}{2}(F_{r-1}(H_n H_{n+1}^2 - D(-1)^n H_{n-1}) + F_r H_n H_{n+1} H_{n+2}) \tag{1.18}$$

To complete the generalization we require, in addition to the result just derived,

$$\Sigma H_n H_{n+1} H_{n+r}$$

Now:

$$\begin{aligned} H_n H_{n+1} H_{n+r} &= H_n H_{n+1}(F_{r-1} H_n + F_r H_{n+1}) \\ &= F_{r-1} H_n^2 H_{n+1} + F_r H_n H_{n+1}^2 \end{aligned}$$

Using (1.18)

$$\begin{aligned} \Sigma H_n H_{n+1} H_{n+r} &= \tfrac{1}{2} F_{r-1} H_n H_{n+1} H_{n+2} \\ &+ \tfrac{1}{2} F_r (H_{n+1}^2 H_{n+2} - D(-1)^n H_n) \end{aligned} \tag{1.19}$$

All that remains now is to combine (1.18) and (1.19) in the same sort of way.

$$\begin{aligned} 2\Sigma H_n H_{n+r} H_{n+s} &= F_{s-1} F_{r-1}(H_n H_{n+1}^2 - D(-1)^n H_{n-1}) + F_{s-1} F_r H_n H_{n+1} H_{n+2} \\ &+ F_s F_{r-1} H_n H_{n+1} H_{n+2} + F_s F_r (H_{n+1}^2 H_{n+2} - D(-1)^n H_n) \end{aligned} \tag{1.20}$$

Concentrating for the moment on the last term; this is:

$$\begin{aligned} F_s F_r (H_{n+1}^2 H_{n+2} - D(-1)^n (H_{n+1} - H_{n-1})) &= F_s F_r (H_{n+1}^2 H_{n+2} + D(-1)^n H_{n-1} \\ &+ H_{n+1}(H_n H_{n+2} - H_{n+1}^2)) \end{aligned}$$

Substituting this in (1.20) we have:

$$2\Sigma H_n H_{n+r} H_{n+s} = (F_s F_r - F_{s-1} F_{r-1}) D(-1)^n H_{n-1}$$
$$+ (F_{s-1} F_{r-1} + F_s F_r) H_n H_{n+1}^2$$
$$+ (F_s F_r + F_s F_{r-1} + F_{s-1} F_r) H_n H_{n+1} H_{n+2}$$

and this simplifies down to:

$$2\Sigma H_n H_{n+r} H_{n+s} = (F_s F_r - F_{s-1} F_{r-1}) D(-1)^n H_{n-1} + H_{s+r+n+1} H_n H_{n+1}$$

(1.21)

PUTTING IN THE LIMITS

We end by quoting the generalized summations with limits from 1 to n.

(2.1) $$\sum_{r=1}^{n} a^r H_{r+s} = \frac{a}{a^2 + a - 1} (a^{n+1}(H_{n+s} - H_s) + a^n (H_{n+s+1} - H_{s+1}))$$

provided $a^2 + a - 1 \neq 0$.

(2.2) $$\sum_{r=1}^{n} r^k H_{r+s} = A_k(n) H_{n+s+2} + B_k(n) H_{n+s+3} - A_k(0) H_{s+2} - B_k(0) H_{s+3} ,$$

where $A_k(n)$, $B_k(n)$ can be generated from (1.11).

(2.3) $$\sum_{r=1}^{n} H_r H_{r+s} = \tfrac{1}{2}(F_{s-3}(H_n H_{n+1} - H_0 H_1) + F_s (H_{n+2}^2 - H_2^2))$$

(2.4) $$\sum_{r=1}^{n} H_r H_{r+s} H_{r+t} = \tfrac{1}{2}(D(F_s F_t - F_{s-1} F_{t-1})((-1)^n H_{n-1} - H_{-1})$$
$$+ H_{s+t+n+1} H_n H_{n+1} - H_{s+t+1} H_0 H_1)$$

THE POLYNOMIALS A AND B

Let

$$X_k(n) = a_0 + a_1 n + \cdots + a_p n^p + \cdots + a_q n^q \, .$$

The table below gives the coefficients a_p of the polynomials A_k, B_k .

$X_k(n)$	a_0	a_1	a_2	a_3	a_4	a_5
A_0	1	0	0	0	0	0
B_0	0	0	0	0	0	0
A_1	0	1	0	0	0	0
B_1	-1	0	0	0	0	0
A_2	2	0	1	0	0	0
B_2	3	-2	0	0	0	0
A_3	-12	6	0	1	0	0
B_3	-19	9	-3	0	0	0
A_4	98	-48	12	0	1	0
B_4	129	-76	18	-4	0	0
A_5	-870	490	-120	20	0	1
B_5	-1501	795	-190	30	-5	0

REFERENCES

1. For a different symbolism and slightly different definition see "Finite Difference Equations," Levy and Lessman, Pitman, London, 1959.
2. Solution to H-17, Erbacher and Fuchs, Fibonacci Quarterly, Vol. 2 (1964), No. 1, p. 51.
3. Solution to B-29, Parker, Fibonacci Quarterly, Vol. 2 (1964), No. 2, p. 160.

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