ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-136 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California, and D. A. Lind, University of Virginia, Charlottesville, Virginia.

Let $\{H_n\}$ be defined by $H_1 = p$, $H_2 = q$, $H_{n+2} = H_{n+1} + H_n$ $(n \geq 1)$, where p and q are non-negative integers. Show there are integers N and k such that $F_{n+k} < H_n \leq F_{n+k+1}$ for all $n > N$. Does the conclusion hold if p and q are allowed to be non-negative reals instead of integers?

H-137 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

GENERALIZED FORM OF H-70: Consider the set S consisting of the first N positive integers and choose a fixed integer k satisfying $0 < k \leq N$. How many different subsets A of S (including the empty subset) can be formed with the property that $a' - a'' \neq k$ for any two elements a', a'' of A: that is, the integers i and $i + k$ do not both appear in A for any $i = 1, 2, \cdots, N - k$.

H-138 Proposed by George E. Andrews, Pennsylvania State University, University Park, Pa.

If F_n denotes the sequence of polynomials $F_1 = F_2 = 1$, $F_n = F_{n-1} + x^{n-2}F_{n-2}$, prove that $1 + x + x^2 + \cdots + x^{p-1}$ divides F_{p+1} for any prime $p \equiv \pm 2 \pmod 5$.

H-139 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$A_n = \begin{bmatrix} F_n & F_{n+1} & \cdots & F_{n+k-1} \\ F_{n+k-1} & F_n & \cdots & F_{n+k-2} \\ \cdot & \cdot & \cdot & \cdot \\ F_{n+1} & F_{n+2} & \cdots & F_n \end{bmatrix},$$

$$M = \begin{bmatrix} A_n & A_{n+k} & \cdots & A_{n+(m-1)k} \\ A_{n+(m-1)k} & A_n & \cdots & A_{n+(m-2)k} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n+k} & A_{n+2k} & \cdots & A_n \end{bmatrix}.$$

Evaluate det M .

For $m = k = 2$ the problem reduces to H-117 (Fibonacci Quarterly, Vol. 5, No. 2 (1967), p. 162).

H-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia

For a positive integer m, let $\alpha = \alpha(m)$ be the least positive integer such that $F_\alpha \equiv 0 \pmod{m}$. Show that the highest power of a prime p dividing $F_1F_2 \cdots F_n$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{\alpha(p^k)}\right],$$

where $[x]$ denotes the greatest integer contained in x. Using this, show that the Fibonacci binomial coefficients

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \qquad (r > 0)$$

are integers.

H-141 Proposed by H. T. Leonard, Jr., and V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

$$\text{(a)}\qquad \frac{F_{3n} + 2^n F_n}{2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} L_{2(n-(2k+1))} F_{2k+1}$$

$$\text{(b)}\qquad \frac{L_{2n} - L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k+1} L_{2k+1}$$

$$\text{(c)}\qquad \frac{L_{2n} + L_n}{2} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} L_{2k}$$

H-142 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

With the usual notation for Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, show that

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \sum_{k=0}^{n} \binom{\frac{1+\sqrt{5}}{1-\sqrt{5}}k}{k} \binom{n - \frac{1+\sqrt{5}}{1-\sqrt{5}}k}{n-k} = F_{n+1} ,$$

where

$$\binom{x}{j} = x(x-1)(x-2)\cdots(x-j+1)/j!$$

is the usual binomial coefficient symbol.

SOLUTIONS

ORIGINAL COMPOSITION

H-88 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California (Corrected).

Prove that

$$\sum_{k=0}^{n} F_{4mk} \binom{n}{k} = L_{2m}^{n} F_{2mn}$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let

$$S = \sum_{k=0}^{n} F_{4mk} \binom{n}{k}$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (p^{4m})^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (q^{4m})^k$$

$$= \frac{1}{\sqrt{5}} \left[(1 + p^{4m})^n - (1 + q^{4m})^n \right]$$

where

$$p + q = 1, \quad pq = -1; \quad \text{or} \quad (pq)^{2m} = 1$$

Hence,

$$S = \frac{1}{\sqrt{5}} \left[\left\{ (pq)^{2m} + p^{4m} \right\}^n - \left\{ (pq)^{2m} + q^{4m} \right\}^n \right]$$

$$= \frac{1}{\sqrt{5}} \left[p^{2mn} (p^{2m} + q^{2m})^n - q^{2mn} (p^{2m} + q^{2m})^n \right]$$

$$= (p^{2m} + q^{2m})^n \frac{(p^{2mn} - q^{2mn})}{\sqrt{5}}$$

$$= L_{2m}^{n} F_{2mn}$$

Therefore,

$$\sum_{k=0}^{n} F_{4mk}\binom{n}{k} = L_{2m}^{n}F_{2mn}$$

Also solved by John Wessner, L. Carlitz, and F. D. Parker.

FINE BREEDING

H-96 Proposed by Maxey Brooke, Sweeny, Texas, and V. E. Hoggatt, Jr., San Jose State College, San Jose, California (Corrected).

Suppose a female rabbit produces $F_n(L_n)$ female rabbits at the n^{th} time point and her female offspring follow the same birth sequence, then show that the new arrivals, C_n, (D_n) at the n^{th} time point satisfies

$$C_{n+2} = 2C_{n+1} + C_n\ ;\ C_1 = 1,\ C_2 = 2$$

and

$$D_{n+1} = 3D_n + (-1)^{n+1}$$

Solution by Douglas Lind, University of Virginia.

Hoggatt and Lind ["The Dying Rabbit Problem," to appear, Fibonacci Quarterly] have proved the following result: Let a female rabbit produce B_n female rabbits at the n^{th} time point, her offspring do likewise, and put

$$B(x) = \sum_{n=1}^{\infty} B_n x^n .$$

Then the number R_n of new arrivals has the generating function

$$R(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{1}{1 - B(x)} ,$$

where we use the convention that $R_0 = 1$ (the original female being born at the 0^{th} time point). We apply this result to the cases (i) $B_n = F_n$, and (ii) $B_n = L_n$.

(i) If $B_n = F_n$, then

$$B(x) = \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} ,$$

so

$$R(x) = 1 + \frac{x}{1 - 2x - x^2} .$$

It is clear from the generating function that here the $R_n = C_n$ obey the recurrence relation $C_{n+2} = 2C_{n+1} + C_n$ along with $C_1 = 1$, $C_2 = 2$, thus establishing the desired result.

(ii) The recurrence relation proposed is incorrect, the proper one being shown below. If $B_n = L_n$, then

$$B(x) = \sum_{n=1}^{\infty} L_n x^n = \frac{x + 2x^2}{1 - x - x^2} ,$$

so that

$$R(x) = \frac{1}{1 - \dfrac{x + 2x^2}{1 - x - x^2}} = 1 + \frac{x + 2x^2}{1 - 2x - 3x^2} .$$

Now

$$\frac{x + 2x^2}{1 - 2x - 3x^2} = -\frac{2}{3} + \frac{\frac{5}{12}}{1 - 3x} + \frac{\frac{1}{4}}{1 + x} .$$

so that

$$D_n = R_n = (5/12)(3^n) + (1/4)(-1)^n \quad (n \geq 1).$$

It follows that $D_1 = 1$, and that $D_{n+1} = 3D_n + (-1)^{n+1}$, the correct relation.

BINOMIAL, ANYONE?

H-97 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show

$$\text{(a)} \qquad \sum_{k=0}^{n} \binom{n}{k}^2 L_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{n-k}$$

$$\text{(b)} \qquad \sum_{k=0}^{n} \binom{n}{k}^2 F_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{n-k} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

If

$$P(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

and

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (x-1)^{n-k} ,$$

then $P(x) = Q(x)$ is a known identity (see elementary problem E799, American

Math. Monthly, 1948, p. 30). If α and β are roots of $x^2 - x - 1 = 0$, then $L_n = \alpha^n + \beta^n$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and thus

$$\text{(a)} \qquad P(\alpha) + P(\beta) = Q(\alpha) + Q(\beta) ,$$

since

$$\begin{aligned}(\alpha - 1)^{n-k} + (\beta - 1)^{n-k} &= (\alpha\beta^2)^{n-k} + (\beta\alpha^2)^{n-k} \\ &= (-1)^{n-k} L_{n-k} ,\end{aligned}$$

$$\text{(b)} \qquad (P(\alpha) - P(\beta))/\sqrt{5} = (Q(\alpha) - Q(\beta))/\sqrt{5}$$

since

$$\begin{aligned}(\alpha - 1)^{n-k}(\beta - 1)^{n-k} &= (-1)^{n-k}(\beta^{n-k} - \alpha^{n-k}) \\ &= -(-1)^{n-k}(\sqrt{5})F_{n-k}\end{aligned}$$

PRODUCTIVE SUMS

H-99 Proposed by Charles R. Wall, Harker Heights, Texas.

Using the notation of H-63 (April 1965 FQJ, p. 116), show that if $\alpha = (1 + \sqrt{5})/2$,

$$\prod_{n=1}^{m} \sqrt{5} F_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n-1)/2} F(n, m) \alpha^{-n(m+1)}$$

$$\prod_{n=1}^{m} L_n \alpha^{-n} = 1 + \sum_{n=1}^{m} (-1)^{n(n+1)/2} F(n, m) \alpha^{-n(m+1)} ,$$

where

$$F(n, m) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_1 F_2 \cdots F_n} .$$

Solution by Douglas Lind, University of Virginia.

We use the familiar identity

$$(\star) \qquad \prod_{n=0}^{m-1} (1 - q^n x) = \sum_{n=0}^{m} (-1)^n q^{n(n-1)/2} \begin{bmatrix} m \\ n \end{bmatrix} x^n ,$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-n+1})}{(1-q)(1-q^2)\cdots(1-q^n)} .$$

If $\beta = (1-\sqrt{5})/2$, then $\sqrt{5}F_n\alpha^{-n} = 1 - (\beta/\alpha)^n$. Putting $q = \beta/\alpha$, then

$$\begin{bmatrix} m \\ n \end{bmatrix} = \alpha^{n^2-mn}F(n,m) ,$$

and putting $x = q$ in (*) gives

$$\prod_{n=1}^{m} \sqrt{5}F_n\alpha^{-n} = \prod_{n=1}^{m} (1-q^n) = \sum_{n=0}^{m} (-1)^n F(n,m) q^{n(n+1)/2}\alpha^{n^2-mn}$$

$$= \sum_{n=0}^{m} (-1)^{n(n-1)/2} F(n,m)\alpha^{-n(m+1)}$$

where we have used $\alpha\beta = -1$.

Similarly, $L_n\alpha^{-n} = 1 - (\beta/\alpha)^m$, so putting $q = \beta/\alpha$ and $x = -q$ in (*) gives

$$\prod_{n=1}^{m} L_n\alpha^{-n} = \prod_{n=1}^{m} (1+q^n) = \sum_{n=0}^{m} (-1)^n F(n,m) q^{n(n-1)/2}\alpha^{n^2-mn}(-q)^n$$

$$= \sum_{n=0}^{m} F(n,m) q^{n(n+1)/2}\alpha^{n^2-mn}$$

$$= \sum_{n=0}^{m} (-1)^{n(n+1)/2} F(n,m)\alpha^{-n(m+1)}$$

Also solved by M. N. S. Swamy.

PYTHAGOREANS AND ALL THAT STUFF

H-101 Proposed by Harlan Umansky, Cliffside Park, N. J., and Malcolm Tallman, Brooklyn, N. Y.

Let a, b, c, d be any four consecutive generalized Fibonacci numbers (say $H_1 = p$ and $H_2 = q$ and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$), then show

$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

Let $A = L_k L_{k+3}$, $B = 2L_{k+1}L_{k+2}$, and $C = L_{2k+2} + L_{2k+4}$. Then show

$$A^2 + B^2 = C^2 \quad .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Now

$$\begin{aligned}(cd - ab)^2 &= [c(b + c) - b(c - b)]^2 \\ &= (c^2 + b^2)^2 = (c^2 - b^2)^2 + (2bc)^2 \\ &= (c + b)^2(c - b)^2 + (2bc)^2 = d^2a^2 + (2bc)^2\end{aligned}$$

Hence

$$(1) \qquad (cd - ab)^2 = (ad)^2 + (2bc)^2$$

Since L_k, the Lucas number, is also a generalized Fibonacci sequence with

$$L_1 = p = 1, \quad L_2 = q = 3,$$

we have that for the four consecutive Lucas numbers L_k, L_{k+1}, L_{k+2}, L_{k+3},

$$(2) \qquad (L_{k+2}L_{k+3} - L_kL_{k+1})^2 = (L_kL_{k+3})^2 + (2L_{k+1}L_{k+2})^2 = A^2 + B^2$$

Now

$$\begin{aligned}
(L_{k+2}L_{k+3} - L_kL_{k+1}) &= L_{k+2}(L_{k+2} + L_{k+1}) - L_{k+1}(L_{k+2} - L_{k+1}) \\
&= L_{k+2}^2 + L_{k+1}^2 = (F_{k+3} + F_{k+1})^2 + (F_{k+2} + F_k)^2 \\
&= (F_{k+2} + 2F_{k+1})^2 + (2F_{k+2} - F_{k+1})^2 \\
(3) \qquad &= 5(F_{k+2}^2 + F_{k+1}^2) = 5F_{2k+3} \\
&= 2F_{2k+3} + (F_{2k+5} - F_{2k+4}) + (F_{2k+2} + F_{2k+1}) + F_{2k+3} \\
&= (F_{2k+3} + F_{2k+1}) + (F_{2k+5} + F_{2k+3}) \\
&= L_{2k+2} + L_{2k+4} = C
\end{aligned}$$

Thus, from (2) and (3) we have,

$$A^2 + B^2 = C^2 .$$

Also solved by J. A. H. Hunter and A. G. Shannon.

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[Continued from p. 285]

RECURRING SEQUENCES — LESSON 1

ANSWERS TO PROBLEMS

1. $a_n = n(n + 1)$; $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
2. $a_n = 3n - 2$; $T_{n+2} = 2T_{n+1} - T_n$
3. $a_n = n^3$; $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
4. $T_{6n+k} = 1, 3, 3, 1, 1/3, 1/3$, for $k = 1, 2, 3, 4, 5, 6$, respectively
5. $T_{n+1} = \sqrt{1 + T_n^2}$
6. $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
7. $T_{n+1} = aT_n$
8. $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
9. $T_{2n-1} = a$, $T_{2n} = 1/a$
10. $T_{n+1} = 1/(2 - T_n)$

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