A NEW ANGLE ON PASCAL'S TRIANGLE

V. E. HOGGATT, JR.
San Jose State College, San Jose, Calif.

1. INTRODUCTION

There has always been such interest in the numbers in Pascal's arithmetic triangle. The sums along the horizontal rows are the powers of two, while the sums along the rising diagonals are the Fibonacci numbers. An early paper by Melvin Hochster [6] generalized the Fibonacci number property by using the left-justified Pascal Triangle and taking other diagonal sums, the first summand being a one on the left edge and subsequent summands are obtained by moving p units up and q units to the right until one is out of the triangle. Unfortunately, he required that $(p,q) = 1$. Harris and Styles [4] produced a generalization of these concepts, and yet a further generalization [5]. We present here a simplifying principle which will make the study of generalizations such as those of Lind [8] easier.

2. COLUMN GENERATORS

Consider the columns of binomial coefficients in the left-justified Pascal Triangle shown in (1). The generating functions for these columns of

$$
(1) \qquad \left\{
\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \\
\dfrac{1}{(1-x)} & \dfrac{x}{(1-x)^2} & \dfrac{x^2}{(1-x)^3} & \dfrac{x^3}{(1-x)^4} & \text{Column Generator} \\
0^{\text{th}} & 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & \text{Column}
\end{array}
\right.
$$

(Received March, 1966)

coefficients, indicated in (1), are given by the corresponding Maclaurin series. That is,

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \binom{n}{0} x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} \binom{n}{1} x^n$$

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \sum_{n=0}^{\infty} \binom{n}{2} x^n$$

$$\vdots$$

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

where we have used the usual convention that

$$\binom{n}{k} = 0$$

for $n < k$. We should note that the column generators

$$g_k(x) = \frac{x^k}{(1-x)^{k+1}} \quad (k = 0, 1, 2, \cdots)$$

automatically align the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$$

as the coefficients of x^n. Using the above generators, the generating function for the sum of the binomial coefficients across the n^{th} row of Pascal's Triangle is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k = \frac{1}{(1-x)\left(1 - \frac{x}{1-x}\right)}$$

$$= \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n .$$

This yields the familiar identity

$$\sum_{j=0}^{\infty} \binom{n}{j} = 2^n .$$

If, on the other hand, we multiply each generating function $g_k(x)$ by λ^k and sum again, we find

$$G(x, \lambda) = \sum_{k=0}^{\infty} \lambda^k g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{\lambda x}{1-x}\right)^k$$

$$= \frac{1}{1-(1+\lambda)x} = \sum_{n=0}^{\infty} (1+\lambda)^n x^n ,$$

Thus by equating coefficients of x^n in each representation we get

$$\sum_{j=0}^{n} \binom{n}{j} \lambda^n = (1+\lambda)^n .$$

If we multiply the generating function $g_k(x)$ by appropriate powers of x, this allows us to vertically shift the separate columns, aligning the numbers along certain diagonals in a horizontal row.

3. THE RISING DIAGONAL SUMS

If we wish to sum the numbers along the rising diagonals, we modify the column generators to be

$$g_k^\star(x) = \frac{x^{2k}}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n-k}{k} x^n \text{ .}$$

The diagonal sums, derived from (1), arė displayed with appropriate column generating functions in (2). We now obtain a generating function

$$(2) \qquad \left\{ \begin{array}{llllll}
1 & 1 & & & & \\
1 & 1 & & & & \\
2 & 1 \text{———} & 1 & & & \\
3 & 1 \text{———} & 2 & & & \\
5 & 1 \text{———} & 3 \text{———} & 1 & & \\
8 & 1 \text{———} & 4 \text{———} & 3 & & \\
13 & 1 \text{———} & 5 \text{———} & 6 \text{———} & 1 & \\
\cdot & \cdot & & \cdot & & \\
\cdot & \cdot & & \cdot & & \\
\cdot & \cdot & & \cdot & & \\
F_n & \binom{n}{0} & \binom{n-1}{1} & \binom{n-2}{2} & \binom{n-3}{3} & \cdots \\
 & \dfrac{1}{1-x} & \dfrac{x^2}{(1-x)^2} & \dfrac{x^4}{(1-x)^3} & \cdots & \text{Column Generators}
\end{array} \right.$$

for the sums of the n^{th} row,

$$G(x) = \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x} \right)^k = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n \text{ ,}$$

a well known result.

4. GENERALIZED FIBONACCI NUMBERS

We now turn to the first generalization of the Fibonacci numbers due to Hochster [6] and Harris and Styles [4]. These numbers are given by

$$u(n;p,q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} \qquad (n \geq 0),$$

where $[x]$ denotes the greatest integer $\leq x$. In particular, $u(n;1,1) = F_{n+1}$ and $u(n;0,1) = 2^n$. To get these sums from the left-adjusted Pascal Triangle we form sums beginning with the $(n+1)^{st}$ one in the leftmost column and add all the coefficients obtained by moving p units up and q units to the right until out of the Triangle. The column generators which yield such summands in a horizontal line are

$$g_k(x) = \frac{x^{k(p+q)}}{(1-x)^{kq+1}} \quad .$$

Thus

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} g_k(x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \left\{ \frac{x^{p+q}}{(1-x)^q} \right\}^k \\ &= \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;p,q)x^n \quad (p+q \geq 1;\ q \geq 0). \end{aligned} \tag{3}$$

This generating function was not given in [4], but is a special case of one given in [9]. We note that in (3) p may be negative. If $p = 1$ and $q = 1$, then (3) becomes

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n ,$$

while for $q = 2$ and $p = -1$ we have

$$\frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1}x^n ,$$

so that there are also Fibonacci numbers in the falling diagonals.

5. A FURTHER GENERALIZATION

In a new paper [5], Harris and Styles consider Pascal's Triangle with each row repeated s times. The column generators for the new array can be easily obtained. The column generator

$$g_k(x) = \frac{1}{(1 - x)^{k+1}}$$

generates the coefficients in the k^{th} column of a left-adjusted Pascal Triangle, and

$$h_k(x) = \frac{1}{(1 - x^s)^{k+1}}$$

has the same coefficients as $g_k(x)$, except each nonzero entry is separated by $s - 1$ consecutive zeros. We can modify the $h_k(x)$ to duplicate each nonzero entry s times by multiplying it by $1 + x + x^2 + \cdots + x^{s-1}$. Thus

$$h_k^*(x) = \frac{1 + x + \cdots + x^{s-1}}{(1 - x^s)^{k+1}} = \frac{1}{(1 - x)(1 - x^s)^k} .$$

To align the coefficients of like powers of x requires

$$g_k^\star(x) = \frac{x^{ks}}{(1-x)(1-x^s)^k} .$$

More generally, if we are interested in summing as before by going along rising diagonals in steps of p units up and q units to the right (see Section 4), then the required column generators will become

$$g_k^\star(x) = \frac{x^{k(p+sq)}}{(1-x)(1-x^s)^{kq}} .$$

The generating function for the numbers

$$u(n;p,q,s) = \sum_{k=0}^{\left[\frac{n}{p+sq}\right]} \binom{\frac{n-pk}{s}}{qk} \qquad (n \geq 0)$$

investigated in [5] is thus

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{1-x}\sum_{k=0}^{\infty}\left(\frac{x^{p+sq}}{(1-x^s)^q}\right)^k \\ &= \frac{(1-x^s)^q/(1-x)}{(1-x^s)^q - x^{p+sq}} = \sum_{n=0}^{\infty} u(n;p,q,s)\,x^n . \end{aligned} \tag{4}$$

The horizontal sums will be finite if $p + sq \geq 1$, $s > 0$, and $q \geq 0$, so again p may be negative. For example, if $p = -1$, $q = 1$, and $s = 2$, then

$$G(x) = \frac{(1-x^2)/(1-x)}{(1-x^2)-x} = \frac{1+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+2}\,x^n ,$$

so there are Fibonacci numbers even in the falling diagonals of the left-adjusted Pascal Triangle with each row repeated s times.

6. THE TRIMMED PASCAL TRIANGLE

Let us return to the numbers $u(n;p,q)$ of [4] (see Section 4). Suppose we define $u^\star(n;p,q)$ as having the same summation pattern (p units up and q units to the right), but in Pascal's Triangle with the first m columns removed. Letting $g_k^\star(x)$ be the generating function for the k^{th} column of this trimmed, left-justified Pascal Triangle, it easily follows that

$$g_k^\star(x) = \frac{x^{k(p+q)}}{(1-x)^{m+1+kq}} .$$

Therefore the generating function for the numbers $u^\star(n;p,q)$ is

$$G^\star(x) = \sum_{k=0}^{\infty} g_k^\star(x) = \frac{1}{(1-x)^{m+1}} \sum_{k=0}^{\infty} \left(\frac{x^{p+q}}{(1-x)^q}\right)^k = \frac{1}{(1-x)^m} \cdot \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} .$$

We point out that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

then

$$\frac{f(x)}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j\right) x^n ,$$

so that multiplying the generating function for the $u(n;p,q)$ by $(1-x)^{-1}$ merely yields the generating function for the partial sums of the $u(n;p,q)$. Repeated application m times yields m-fold partial sums. Thus we note if we take rising diagonals on Pascal's Triangle with the left column of ones trimmed off, the result will be the sum of the Fibonacci numbers, so that

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 ,$$

while consideration of row sums gives

$$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$$

(see Figure 1). In general we have

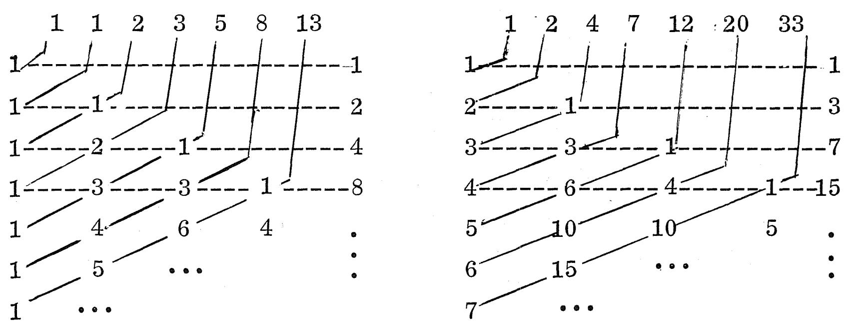

Figure 1

$$\sum_{k=0}^{n} u(k;p,1) = u(n+p+1;\ p,1) - 1 .$$

We also note that the original generating function for the Fibonacci numbers,

$$G(x) = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n ,$$

becomes

$$G^\star(x) = \frac{1}{(1-x)^m} \cdot \frac{1}{1-x-x^2}$$

for Pascal's Triangle trimmed of the first m columns. Thus we have

$$G^\star(x) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{m+k-1}{m-1} F_{n-k} \right\} x^n ,$$

a convolution of the Fibonacci numbers with the $(m-1)^{st}$ column of Pascal's Triangle. If the column of ones is deleted, so that $m = 1$, the generating function for $p = -1$ and $q = 2$ is

$$G(x) = \frac{1}{1-x} \cdot \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+2}\, x^n ,$$

so Fibonacci numbers are again in the falling diagonals.

Returning to the general case of the generating function for the $u(n;p,q)$ given in (3), we remark that in this particular case we can interpret the sequence generated by

$$\frac{(1-x)^{q-1-m}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u^\star(n;p,q)\, x^n \quad (m = 0, 1, \cdots) .$$

7. A SURPRISE CONNECTION

In an important paper concerning unique representations of the positive integers as sums of distinct Fibonacci numbers and the generalization of this representation property, D. E. Daykin ([1], [2], [3]) studied the sequence

$$\{u_n\}_{n=0}^{\infty}$$

defined by

$$\begin{cases} u_n = n \quad (n = 1, 2, 3, \cdots, r) , \\ u_n = u_{n-1} + u_{n-r} \quad (n > r) . \end{cases}$$

Now if $r = 1$, we get $u_n = 2^{n-1}$, while if $r=2$, then $u_n = F_{n+1}$. Consider the generating function for the numbers $u(n;\ r-1, 1)$,

$$\frac{1}{1-x-x^r} = \sum_{n=0}^{\infty} u(n;\ r-1,\ 1)\, x^n .$$

The initial values are $u(n; r-1, 1) = 1$ for $n = 0, 1, \cdots, r-1$, and $u(n; r-1, 1) = n+2-r$ for $n = r,\ r+1,\ \cdots,\ 2r-1$. Thus

$$u_n = u(n+r-2;\ r-1,\ 1)$$

for $n \geq 1$. Hence the generating function for the u_n is

$$\sum_{n=1}^{\infty} u(n+r;\ r-1,\ 1)x^n = \left[\frac{1}{1-x-x^r} - (1 + x + \cdots + x^{r-2})\right] \Big/ x^{r-1}$$

$$= \frac{(1-x^r)\ /\ (1-x)}{1-x-x^r}$$

But this is a special case of (4). Thus the second generalized Fibonacci numbers $u(n;p,q,s)$ of Harris and Styles reduce to the u_n by choosing $s = r$, $q = 1$, and $p = r-1$.

D. E. Daykin also studied ([1], [2], [3]) the sequence

$$\{v_n\}_{n=1}^{\infty}$$

defined by

$$\begin{cases} v_n = n & (n = 1,\ 2,\ \cdots,\ r), \\ v_n = v_{n-1} + v_{n-r} + 1 & (n > r) . \end{cases}$$

It can be easily verified that the numbers $u\ (n+r-2;\ r-1, 1)$, summed in Pascal's Triangle with the first column deleted, obey the same recurrence relation and boundary conditions as the v_n, so that $v_n = u^\star(n + r - 2; r - 1, 1)$ for $n \geq 1$. Thus the generating function for the v_n is

$$\sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} u^\star(n + r - 2; r - 1, 1)x^n = \frac{1}{(1 - x)(1 - x - x^r)}$$

8. SOME FURTHER RESULTS

Let $f(x)$ be the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Suppose we multiply each of the column generators $g_k(x)$ by the corresponding coefficient a_k and sum, yielding

$$G(x) = \sum_{k=0}^{\infty} a_k g_k(x) .$$

In many particular cases the results are quite interesting. For example, let

$$f(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n ,$$

$$g_k(x) = \frac{x^k}{(1 - x)^{k+1}} .$$

Then

$$G(x) = \sum_{k=0}^{\infty} F_{k+1} \frac{x^k}{(1 - x)^{k+1}} = \frac{1}{1 - x} \sum_{k=0}^{\infty} F_{k+1} \left(\frac{x}{1 - x}\right)^k$$

$$= \frac{1}{(1 - x)\left(1 - \frac{x}{1 - x} - \frac{x^2}{(1 - x)^2}\right)} = \frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

Since in this case

$$g_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

we have that

$$\sum_{k=0}^{n} F_{k+1} \binom{n}{k} = F_{2n+1} .$$

If, on the other hand, we put

$$g_k(x) = \frac{x^{2k}}{(1-x)^{k+1}} ,$$

then we are multiplying F_{k+1} by the corresponding elements of the rising diagonals, and

$$G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \binom{n-k}{k} F_{k+1} \right) x^n = \frac{1}{1-x} \sum_{k=0}^{\infty} F_{k+1} \left(\frac{x^2}{1-x} \right)^k$$

$$= \frac{1-x}{1-2x+x^3-x^4} .$$

Suppose that

$$f(x) = \frac{1-x}{1-2x-2x^2+x^3} = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

Then

$$G(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\binom{n-k}{k}F_{k+1}^2\right)x^n = \frac{1}{1-x}\sum_{k=0}^{\infty}F_{k+1}^2\left(\frac{x}{1-x}\right)^k$$

$$= \frac{1 - \frac{x}{1-x}}{(1-x)\left(1 - 2\frac{x}{1-x} - 2\frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3}\right)}$$

$$= \frac{(1-2x)(1-x)}{1-5x-5x^2} .$$

There are thus many easily accessible generating functions where the numbers generated are multiplied by the corresponding elements on any of the diagonals whose sums are the $u(n;p,q)$. These methods were discussed in [7].

REFERENCES

1. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers," Journal London Math. Soc., 35 (1960), 143-160.
2. D. E. Daykin and A. J. W. Hilton, "Bases for Infinite Intervals of Integers," Fibonacci Quarterly, Vol. 5, No. 4, Nov., 1967, pp. 329-346.
3. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers — II," to appear, Fibonacci Quarterly.
4. V. C. Harris and Carolyn C. Styles, " A Generalization of Fibonacci Numbers," Fibonacci Quarterly, 2 (1964), 277-289.
5. V. C. Harris and Carolyn C. Styles, "A Further Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 4, No. 3, pp. 241-248.
6. Melvin Hochster, "Fibonacci-Type Series and Pascal's Triangle," Particle, Vol. IV (1962), 14-28.
7. V. E. Hoggatt, Jr. and D. A. Lind, "Symbolic Substitutions into Fibonacci Polynomials," Fibonacci Quarterly, November, 1968, pp. 55-74.
8. D. A. Lind, "On a Class of Nonlinear Binomial Sums," Fibonacci Quarterly, 3 (1965), 292-298.

* * * *