RECREATIONAL MATHEMATICS

Joseph S. Madachy
4761 Bigger Rd., Kettering, Ohio

Before I go on with new business, readers of this column should make the following corrections in the February 1968 issue of the Fibonacci Quarterly (Vol. 6, No. 1):

Page 64: In 187^{16}, the fifth group of five digits should read 87257 and not 78257.

Page 67: The last few words in the fifth line under "A Fibonacci Variation" should read "$\cdots$ ${}_nF$ series in which each..."

Page 67: Under "Some Fibonacci Queries," for $F_{18} = 2584$, correct the addition to read $2 + 5 + 8 + 4 = 19$.

Some browsing by myself through past issues of the Fibonacci Quarterly disclosed an article by Dewey C. Duncan [2] in which Mr. Duncan anticipated — in a slightly different manner — my Fibonacci variation [4, page 67]. I had formed an ${}_nF$ series in which each term is the sum of the next two terms, starting with ${}_0F = 0$, ${}_1F = 1$:

$$0, \quad 1, \quad -1, \quad 2, \quad -3, \quad 5, \quad -8, \quad 13, \quad -21, \quad 34, \quad -55,$$

etc.

Mr. Duncan introduces Fibonacci number relationships involving zero and negative indices, with

$$F_0 = 0, \quad F_{-1} = 1, \quad F_{-2} = -1, \quad F_{-3} = 2$$

and, generally, $F_{-n} = (-1)^{n+1}F_n$. The Duncan series thus formed is

$$0, \quad 1, \quad -1, \quad 2, \quad -3, \quad 5, \quad -8, \quad 13, \quad -21, \quad 34, \quad -55,$$

etc., which is identical to the ${}_nF$ series given previously.

If n is zero or even, we have $F_{-n} = -F_n$ and $_nF = -F_n$ for n odd, we have $F_{-n} = F_n$ and $_nF = F_n$.

Such is the beauty of the Fibonacci numbers and their variations!

PRODUCTS WITH DIFFERENT FACTORS CONTAINING THE SAME DIGITS

The following collection was derived from the Nelson table described in [4, pp. 61-63]. The list shows products with two sets of factors containing the same digits, e.g. (6)(4592) = (56)(492). Trivial solutions or those derived from simpler forms, are not listed. For example

$$(23)(794) = (23)(794)$$
$$(6)(500) = (600)(5)$$

and others similar to the above are excluded.

The list contains one set of factors (the 8^{th} set) in which the digits are in the same order, and four sets of factors (the first four) in which the digits are in reverse order.

If the list proves incomplete, I would deeply appreciate new results found by readers.

$(Factors)_1$		$(Factors)_2$	Product
(6)(21)	=	126	126
(3)(51)	=	153	153
(50)(6)	=	(60)(5)	300
(4)(567)	=	(7)(6)(54)	2,268
(6)(3128)	=	(23)(816)	18,768
(4)(72)(86)	=	24, 768	24,768
(6)(4592)	=	(56)(492)	27,552
(7)(3942)	=	(73)(9)(42)	27,594
(9)(3465)	=	(63 x 495)	31,185
(53)(781)	=	(71)(583)	41,393
(9)(7128)	=	(81)(792)	64,152
(4)(56)(729)	=	(9)(24)(756)	163,296
(6)(93)(428)	=	(248)(963)	238,824

(7)(52)(918)	=	(9)(51)(728)	334,152
(92)(8736)	=	(96)(8372)	803,712
(6)(7)(84)(531)	=	(8)(413)(567)	1,873,368
(82)(53671)	=	(562)(7831)	4,401,022
(8)(935721)	=	(9)(831752)	7,485,768
(24)(756)(813)	=	(54)(273168)	14,751,072
(9)(76)(25143)	=	(57)(493)(612)	17,197,812
(4)(86)(53217)	=	(216)(84753)	18,306,648
(34)(96)(5721)	=	(576)(32419)	18,673,344
(4)(657)(8213)	=	21,583,764	21,583,764
(9)(561)(4372)	=	(594)(37162)	22,074,228
(64)(78)(9251)	=	(96)(572)(841)	46,180,992

In the April 1968 issue of the Fibonacci Quarterly [5, p. 166], I had asked you to demonstrate that no consecutive set of Fibonacci numbers could be used to form a magic square. In any $n \times n$ (n must be greater than 2) magic square composed of n^2 positive integers, the magic constant (the sum of the integers in each row, column, and long diagonal) is the sum of all the integers divided by n. Therefore, any integer appearing in a magic square must be smaller than the magic constant.

The demonstration involves showing that the largest integer appearing in an array of consecutive Fibonacci numbers is larger than the magic constant — hence such a magic square is impossible.

The sum of the first p Fibonacci numbers is $F_{p+2} - 1$, where F_{p+2} is the $(p + 2)^{th}$ Fibonacci number. The sum of any q consecutive Fibonacci numbers, where F_p is the first and F_{p+q-1} is the last term is

$$(F_{p+q+1} - 1) - (F_{p+1} - 1) = F_{p+q+1} - F_{p+1} .$$

Let F_p be the first integer in a series of n^2 consecutive Fibonacci numbers. The largest will be F_{p+n^2-1} and the sum of these n^2 terms will be

$$F_{p+n^2+1} - F_{p+1} = S_{array} \tag{1}$$

where S_{array}, then, is the sum of the integers in an $n \times n$ array of n^2 consecutive Fibonacci numbers. From equation (1) we can write

(2) $$S_{array} < F_{p+n^2+1}$$

Three consecutive Fibonacci numbers, starting with F_{p+n^2-1} are:

$$F_{p+n^2-1}\,, \quad F_{p+n^2}\,, \quad F_{p+n^2+1}$$

where

$$F_{p+n^2+1} = F_{p+n^2-1} + F_{p+n^2} \;.$$

Also, in any set of three consecutive Fibonacci numbers (excluding the first three 1, 1, 2), we have

$$F_{p+n^2} - F_{p+n^2-1} < F_{p+n^2-1}$$

or

$$F_{p+n^2} = F_{p+n^2-1} + K \;,$$

where

$$K < F_{p+n^2-1} \;.$$

Then

$$F_{p+n^2+1} = F_{p+n^2-1} + F_{p+n^2-1} + K = 2F_{p+n^2-1} + K \;.$$

Since $K < F_{p+n^2-1}$ we have

$$2F_{p+n^2-1} + K < 3F_{p+n^2-1}$$

or

(3) $$F_{p+n^2+1} < 3F_{p+n^2-1} .$$

From inequalities (2) and (3) we have

(4) $$S_{array} < 3F_{p+n^2-1} .$$

If we divide (4) by 3 we obtain

$$\frac{S_{array}}{3} < F_{p+n^2-1} .$$

That is, the magic constant for a 3 x 3 array of 9 consecutive Fibonacci numbers will be less than the largest Fibonacci number in the array. It follows that

$$\frac{S_{array}}{n} < F_{p+n^2-1} , \quad (n < 3)$$

where $(S_{array})/n$ is the magic constant for an n x n array, is also true — and so consecutive Fibonacci numbers cannot be used to construct magic squares.

Some general results concerning Fibonacci numbers and magic squares appear in [1]. There Brown proves the general case that no set of distinct Fibonacci numbers can form a magic square.

Also in [3] Freitag shows a magic square constructed with Fibonacci numbers and sums of Fibonacci numbers. One magic square is shown which has terms, each of which is composed of the sum of two Fibonacci numbers.

This last item raised a trick question which I pass on to readers: Can a magic square be constructed in which each term is the sum of two consecutive Fibonacci numbers?

This column for the December 1968 issue will contain an article by Free Jamison and V. E. Hoggatt, Jr., on the dissection of a square into acute isosceles triangles — an extension of a familiar idea. Also, as a result of some work by Charles W. Trigg appearing in the July 1968 issue of the Journal of Recreational Mathematics, I'll present some recreations in instant division.

REFERENCES

1. John L. Brown, Jr., "Reply to Exploring Fibonacci Magic Squares," Fibonacci Quarterly, Vol. 3, No. 2, April 1965, page 146.
2. Dewey C. Duncan, "Chains of Equivalent Fibonacci-Wise Triangles," Fibonacci Quarterly, Vol. 5, No. 1, February 1967, pp. 87-88.
3. Herta T. Freitag, "A Magic Square involving Fibonacci Numbers," Fibonacci Quarterly, Vol. 6, No. 1, February 1968, pp. 77-80.
4. Joseph S. Madachy, "Recreational Mathematics," Fibonacci Quarterly, Vol. 6, No. 1, February 1968, pp. 60-68.
5. Joseph S. Madachy, "Recreational Mathematics," Fibonacci Quarterly, Vol. 6, No. 2, April 1968, pp. 162-166.

* * * * *

[Continued from p. 287.]

It is also relatively easy to demonstrate that a positive integer n is a perfect number if and only if the sum of the reciprocals of the positive integer divisors of n is 2.

If you have some free time why don't you investigate the topic of perfect numbers or, better yet, why don't you suggest it as a possible project for some talented student in one of your high school mathematics classes?

* * * * *

[Continued from p. 298.]

With these the desired results are immediately available.

Also solved by Herta T. Freitag, C. B. A. Peck, A. C. Shannon (Australia), and the proposer.

* * * * *