A LINEAR ALGEBRA CONSTRUCTED FROM FIBONACCI SEQUENCES
PART I: FUNDAMENTALS AND POLYNOMIAL INTERPRETATIONS

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The purpose of this paper is to demonstrate the construction of a linear algebra with whole Fibonacci sequences as elements. Sequences of complex numbers are considered; hence, this is an algebra over the complex field.

To be of more than curious interest, of course, the algebra must lead somewhere. The vector space leads to geometric interpretation of sequences. The ring leads to polynomial interpretations, and in particular, to binomial expressions. Part II will deal with functions and Taylor series representations.

Only a knowledge of modern algebra at the undergraduate level is required to follow the discussion in Part I. A smattering of topology is required for Part II. Proofs are elementary and are usually based on definitions. In some cases, the reader is asked to fill in the details himself. We begin with:

Definition 1.1. A Fibonacci sequence $U = (u_i)$, $i = 0, 1, \cdots$, is a sequence that has the following properties:

1. u_0, u_1 are arbitrary complex numbers,
2. $u_{n+1} = u_n + u_{n-1}$, $n = 1, 2, \cdots$.

$\mathfrak{F}$ will denote the set of all Fibonacci sequences. Any sequence may be extended to negative subscripts by transposing the recurrence formula; i.e., $u_{n-1} = u_{n+1} - u_n$.

A list of special sequences follows:

$$A = (1, \alpha, \alpha^2, \cdots), \quad \alpha = \frac{1+\sqrt{5}}{2}$$

$$B = (1, \beta, \beta^2, \cdots), \quad \beta = \frac{1-\sqrt{5}}{2}$$

$$F = (0, 1, 1, 2, \cdots)$$

$$I = (1, 0, 1, 1, \cdots)$$

$$L = (2, 1, 3, 4, \cdots)$$

$$O = (0, 0, 0, \cdots)$$

In addition to this, we use the symbols C, R, and Z for the complex, reals, and integers, respectively.

Definition 1.2. For all $U, V \in \mathfrak{F}$, $U = V \Leftrightarrow u_i = v_i$, $i = 0, 1, 2, \cdots$.

Definition 1.3. For $U, V \in \mathfrak{F}$, $U + V = (u_i + v_i)$, $i = 0, 1, \cdots$.

Definition 1.4. For $a \in C$, $U \in \mathfrak{F}$, $aU = (au_i)$, $i = 0, 1, \cdots$.

Theorem 1.1. $\mathfrak{F}$ is a vector space.

Proof. It is a well-known fact that sums and scalar products of Fibonacci sequences yield Fibonacci sequences. The reader is asked to fill in the remainder of the proof from the definition of a vector space. The zero vector is $(0, 0, \cdots)$, and any additive inverse is given by $-U = (-u_0, -u_1, \cdots)$.

Theorem 1.2. The dimension of $\mathfrak{F}$ is 2.

Proof. Consider the vectors I, F, and O, and suppose that $aI + bF = O$. By definitions 2, 3, and 4, the first two terms yield $a = b = 0$. If we insist that a or b be non-zero, then $aI + bF = U \neq O$. We now find that $a = u_0$, $b = u_1$. From $u_0I + u_1F = U$ we find from the n^{th} term that $u_0F_{n-1} + u_1F_n = u_n$, which is a well-known property of all Fibonacci sequences. Hence, an arbitrary vector is uniquely determined by two linearly independent vectors in F, and the theorem is proved.

Theorem 1.3. F is isomorphic to $V_2(C)$, the vector space of all ordered pairs of complex numbers.

Proof. Any vector space is isomorphic to the vector space of n-tuples of its components relative to a fixed basis. Hence, for

$$U \in \mathfrak{F}, \quad U = u_0I + u_1F \Longleftrightarrow U \longleftrightarrow (u_0, u_1) \in V_2(C).$$

As a consequence of Theorem 1.3, we may agree to identify an arbitrary sequence $U = (u_i)$, $i = 0, 1, \cdots$, with the pair (u_0, u_1), and write $U = (u_0, u_1)$. Property 2 of definition 1.1 has been suppressed, so we turn our attention to the construction of a ring that will bring this property back into evidence.

Definition 1.5. For $U, V \in \mathfrak{F}$, $UV = (u_0 v_0 + u_1v_1, u_0v_1 + u_1v_0 + u_1v_1)$.

Theorem 1.4. $\mathfrak{F}$ is a commutative linear algebra with unity $I = (1, 0)$.

Proof. The reader is asked to fill in the details again.

Associated with each sequence is a complex number, called the characteristic number, that describes many properties of the sequence in the algebra.

Definition 1.6. The characteristic number $C(U)$ of a sequence $U = (u_0, u_1)$ is the complex number $u_0^2 + u_0u_1 - u_1^2 = u_0u_2 - u_1^2$.

Theorem 1.5. $C(U) \neq 0 \Leftrightarrow U$ has a multiplicative inverse $U^{-1} \in \mathfrak{F}$.

Proof. If U has an inverse (x, y), then $(u_0, u_1)(x, y) = (1, 0)$. This is equivalent to the equations

$$
\begin{aligned}
u_0 x + u_1 y &= 1 \\
u_0 y + u_1 x + u_1 y &= 0 .
\end{aligned}
\tag{1}
$$

Since either $u_0 \neq 0$ or $u_1 \neq 0$, we may reduce equations 1 to

$$
\begin{aligned}
x(u_0^2 + u_0 u_1 - u_1^2) &= u_0 + u_1 \\
y(u_0^2 + u_0 u_1 - u_1^2) &= -u_1 .
\end{aligned}
\tag{2}
$$

The remainder of the proof is obvious.

Corollary 1.1. If $C(U) \neq 0$, then

$$U^{-1} = \frac{1}{C(U)} (u_2, -u_1) .$$

Corollary 1.2. $C(U) = 0 \Leftrightarrow U = a(1, \alpha),\ b(1, \beta)$.

Proof. Solve the equation $u_0^2 + u_0 u_1 - u_1^2 = 0$ for u_0/u_1 .

The sequence $F = (0, 1)$ plays a major role in the algebra as a shifting operator, and brings property 2 of definition 1.1 back into evidence.

Theorem 1.6. $F^n U = (u_n, u_{n+1}),\ n \in Z$.

Proof. Note that

$$FU = (0, 1)(u_0, u_1) = (u_1, u_0 + u_1),$$

and that

$$F^{-1}U = (-1, 1)(u_0, u_1) = (u_1 - u_0, u_0) .$$

The rest of the proof follows easily by mathematical induction.

Theorem 1.7. $C(F^n U) = (-1)^n C(U),\ n \in Z$.

Proof. Note that

$$
\begin{aligned}
C(FU) &= u_1^2 + u_1 u_2 - u_2^2 = u_1^2 + u_1(u_0 + u_1) - (u_0 + u_1)^2 \\
&= -(u_0^2 + u_0 u_1 - u_1^2) = -C(U) .
\end{aligned}
$$

The rest of the proof follows easily by induction.

Theorem 1.8. $C(U) \neq 0$, and $n \neq m \Leftrightarrow F^nU$, F^mU are linearly independent in $\mathfrak{F}$.

Proof. We test for linear independence by setting $a(u_n, u_{n+1}) + b(u_m, u_{m+1}) = (0, 0)$. This is equivalent to the two equations

$$
\begin{aligned}
au_n + bu_{n+1} &= 0 \\
au_m + bu_{m+1} &= 0 \quad .
\end{aligned}
\tag{3}
$$

Since all $u_i \neq 0$, we may reduce equations 3 to

$$
a(u_n u_{m+1} - u_{n+1} u_m) = 0 \quad . \tag{4}
$$

$n \neq m$ by hypothesis, so let $m = n + k$, and use the identities $u_m = u_{n+k} = u_n F_{k+1} + u_{n+1} F_k$ and $u_{m+1} = u_{n+1+k} = u_{n+1} F_{k-1} + u_{n+2} F_k$. Equation 4 may now be reduced to

$$
a(u_n u_{n+2} - u_{n+1}^2) F_k = aC(F^nU) F_k = 0 \tag{5}
$$

Since $C(F^nU) = (-1)^n C(U) \neq 0$, and $F_k \neq 0$ in general, we must conclude that $a = 0$, which in turn implies that $b = 0$.

The converse is proved by assuming that a, b are not both zero. We can, without loss of generality, assume that $a \neq 0$, which implies that $u_n u_{n+2} - u_{n+1}^2 = 0$. Thus $C(F^nU) = 0 \Rightarrow C(U) = 0$.

An alternate form of the product in $\mathfrak{F}$ is now given.

Theorem 1.9. $UV = u_0 V + u_1 FV = v_0 U + v_1 FU$.

Proof. The proof follows immediately from definition 5.

Multiplication in the ring is equivalent to a linear transformation in the vector space, or symbolically, $UV = U(V) = V(U)$, where $U(V)$ means U transforms V. This can be written in matrix form

$$
UV = (v_0, v_1) \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 + u_1 \end{pmatrix} = (u_0, u_1) \begin{pmatrix} v_0 & v_1 \\ v_1 & v_0 + v_1 \end{pmatrix} \tag{6}
$$

Any sequence of complex numbers can be decomposed into two sequences of real numbers.

Theorem 1.10. $U = X + iY$, where $U \in \mathfrak{F}(C)$, and $X, Y \in \mathfrak{F}(R)$.

Proof. The reader is asked to supply the details.

The vector space in $\mathfrak{F}$ is obviously a unitary 2-space, and the restriction of $\mathfrak{F}$ to real sequences yields a Euclidean 2-space. Some interesting geometric interpretations follow from this, but lack of space prevents further exposition here.

POLYNOMIALS IN $\mathfrak{F}$ OVER C

The polynomial interpretation of $\mathfrak{F}$ leads to some interesting results. We now establish the conditions for writing polynomials with sequences as "indeterminants" and coefficients in the complex field.

Theorem 1.11. C is embedded in $\mathfrak{F}$.

Proof. Let $\psi: C \longrightarrow \mathfrak{F}$ be defined by the rule: $\psi(a) = (a, 0) = aI$, $\forall a \in C$. We ask the reader to complete the proof.

Integral powers of Fibonacci sequences make sense as a consequence of our definition of multiplication in $\mathfrak{F}$. The classic conditions for writing polynomials exist, so that $p(X) = a_0 + a_1X + \cdots + a_nX^n$ makes sense, but this is not the whole story. $p(X)$ is a linear combination of the elements $X^i \in \mathfrak{F}$, and can be expressed uniquely as a linear combination of any two linearly independent elements in $\mathfrak{F}$. If it so happens that $C(X) \neq 0$, then by theorem 1.8, X, FX are linearly independent, and there exist k_0, $k_1 \in C$, not both zero, such that $p(X) = k_0X + k_1FX$. But $K = (k_0, k_1) \in \mathfrak{F}$, and by theorem 1.9, $p(X) = KX$. The linear independence of powers of X does not exist in polynomials in $\mathfrak{F}$ over C. This explains why each of the hundreds (possibly thousands) of known summations involving Fibonacci numbers is expressible as a linear combination of at most two Fibonacci numbers. The addition formula for elements of a Fibonacci sequence is a case in point, which can easily be derived in $\mathfrak{F}$. Try it for an exercise.

The sequences $I^n = I = (1, 0)$ and $F^n = (F_{n-1}, F_n)$ may be written down termwise by inspection. L^n follows easily.

Theorem 1.12. $L^{2k} = 5^k(F_{2k-1}, F_{2k})$, and $L^{2k+1} = 5^k(L_{2k}, L_{2k+1})$.

Proof: $L^2 = (2, 1)(2, 1) = 5(1, 1) = 5F^2$, from which $L^{2k} = 5^kF^{2k}$.

$$L^{2k+1} = L^{2k}L = 5^kF^{2k}L .$$

Several formulas for the general case U^n will be given.

Definition 1.7. The term of U^n bearing the subscript k will be designated $(U^n)_k$, $k = 0, 1, 2, \cdots$ (the k term is actually the $(k+1)^{st}$ term by ordinal count).

Lemma 1.1. Let $c_i \in C$, $i = 0, 1, \cdots, n$, and let $U \in \mathfrak{F}$. Then

$$\sum_{i=0}^{n} c_i U^i = \left(\sum_{i=0}^{n} c_i (U^i)_0 , \sum_{i=0}^{n} c_i (U^i)_1 \right) .$$

Proof. The reader is asked to supply the details.

Theorem 1.13.

$$(U^n)_{k+1} = \sum_{i=0}^{n} \binom{n}{i} u_0^{n-i} u_1^i F_{k+i} .$$

Proof.

$$U^n = (u_0 I + u_1 F)^n = \sum_{i=0}^{n} \binom{n}{i} u_0^{n-i} u_1^i F^i .$$

Lemma 1.1 and definition 1.7 supply the remainder of the proof.

An alternate form of theorem 1.13 is

Theorem 1.14.

$$(U^n)_{k+1} = \frac{\alpha^k (u_0 + \alpha u_1)^n - \beta^k (u_0 + \beta u_1)^n}{\alpha - \beta}$$

Proof. Substitute the Binet formula,

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta} ,$$

into theorem 1.13, and reduce it to the form shown.

Example 1.1. Consider the generating function

$$(7)\qquad D^n = (I + F^k)^n = \sum_{i=0}^{n} \binom{n}{i} F^{ki} \quad ,$$

where

$$(8)\qquad (D^n)_{j+1} = \sum_{i=0}^{n} \binom{n}{i} F_{ki+j} \quad .$$

If $k = 1$, $I + F = (1,0) + (0,1) = (1,1) = F^2$, and

$$(9)\qquad (D^n)_{j+1} = (F^{2n})_{j+1} = F_{2n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \quad .$$

If $k = -1$, $I + F^{-1} = (1,0) + (-1\,,1) = (0,1) = F$, and

$$(10)\qquad (D^n)_{j+1} = (F^n)_{j+1} = F_{n+j} = \sum_{i=1}^{n} \binom{n}{i} F_{-i+j} \quad .$$

But since $F_{-(i-j)} = (-1)^{i-j+1} F_{i-j}$, we have

$$(11)\qquad F_{n+j} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-j+1} F_{i-j} \quad .$$

If $k = 2$, $I + F^2 = (1,0) + (1,1) = (2,1) = L$. From Theorem 1.12, we get

$$(12)\qquad 5^{n/2} F_{n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{2i+j} \text{ , for even n, and } 5^{(n-1)/2} L_{n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{2i+j}, \text{ for odd n.}$$

This may be generalized for even k. If the reader will verify that $I + F^{4m} = L_{2m}F^{2m}$, and $I + F^{4m+2} = F_{2m+1}F^{2m}L$, then he may compute $(D^n)_{j+1}$, and complete the problem.

Much of what we know about polynomials may be applied to polynomials in $\mathfrak{F}$ over C. The possibilities of generating term-by-term Fibonacci relations is unbounded.

ADDITIONAL NOTES

1. Let M be the set of all matrices of the form

$$U = \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 + u_1 \end{pmatrix}, \quad u_0, u_1 \in C,$$

and let the operations be the usual operations of matrix algebra. Then M is isomorphic to F.

2. Let $c[x]$ be the set of polynomials in x over C, and let $s(x) = x^2 - x - 1$. Then $C[x]/s(x)$ is the ring of residue classes of polynomials over C modulo $x^2 - x - 1$. Each residue class has the form $[u_0 + u_1x]$ with operations defined by

$$[u_0 + u_1x] + [v_0 + v_1x] = [u_0 + v_0 + (u_1 + v_1)x]$$

$$[u_0 + u_1x][v_0 + v_1x] = [u_0v_0 + u_1v_1 + (u_0v_1 + u_1v_0 + u_1v_1)x].$$

If we add the redundant operation

$$a[u_0 + u_1x] = [au_0 + au_1x],$$

then $C[x]/s(x)$ is a linear algebra, and furthermore, $C[x]/s(x)$ is isomorphic to $\mathfrak{F}$.

ACKNOWLEDGEMENTS

1. The vector space F is one element of the space of sequences noted by E. D. Cashwell and C. J. Everett, "Fibonacci Spaces," Fibonacci Quarterly, Vol. 4, No. 2, pp. 97-115.

2. The characteristic number of a sequence was used by Brother U. Alfred, "On the Ordering of Fibonacci Sequences," Fibonacci Quarterly, Vol. 1, No. 4, 1963, pp. 43-46. [See Correction, p. 38, Feb. 1964 Quarterly.]

3. A definition for ring multiplication,

$$UV = (u_0v_1 + u_1v_0 - u_1v_1, u_0v_0 + u_1v_1) ,$$

was given by Ken Dill in a paper submitted to the Westinghouse Talent Contest.

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ROMANCE IN MATHEMATICS

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The dome of the famous Taj Mahal, built in 1650 in Agra, India, is ellipsoidal. Now, the ellipse has the geometric property that the angles formed by the focal radii and the normal at a point are congruent. Also, it is a fundamental principle of behavior of sound waves that the angle of incidence equals the angle of reflection. Thus, sound waves issuing from focus A and striking any point on the ellipse will be reflected through focus B.

The builder of the Taj Mahal, Shan Jehan, used these basic principles well in his memorial to his favorite wife who was called Taj Mahal, Crown of the Palace. Honeymooners who visit the shrine are instructed to stand on the two foci which are marked in the tile floor. The husband whispers, "To the memory of an undying love," which can be heard clearly by his wife who is more than fifty feet away but by no one else in the room.

REFERENCE

Kramer, Edna E., "The Mainstream of Mathematics," Premier (paperback), New York, 1961, p. 152.