ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contribution are asked to enclose self-addressed stamped postcards.

B-160 *Proposed by Robert H. Anglin, Dan River Mills, Danville, Virginia.*

Show that, if $x = F_nF_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_{2n+3}$, then $x^2 + y^2 = z^2$.

B-161 *Proposed by John Ivie, Student at University of California, Berkeley, California*

Given the Pell numbers defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$, show that for $k > 0$;

(i) $$P_k = \sum_{r=0}^{[(k-1)/2]} \binom{k}{2r+1} 2^r .$$

(ii) $$P_{2k} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r .$$

B-162 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let r be a fixed positive integer and let the sequences $u_1, u_2, \cdots$ satisfy $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-r}$ for $n > r$ and have initial conditions $u_1 = 2^{j-1}$ for $j = 1, 2, \cdots, r$. Show that every representation of U_n as a sum of distinct u_j must be of the form u_n itself or contain explicitly the terms $u_{n-1}, u_{n-2}, \cdots, u_{n-r+1}$ and some representation of u_{n-r}.

B-163 *Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico*

Let n be a positive integer. Clearly

$$(1 + \sqrt{5})^n = a_n + b_n\sqrt{5} ,$$

with a_n and b_n integers. Show that 2^{n-1} is a divisor of a_n and of b_n.

B-164 *Proposed by J. A. H. Hunter, Toronto, Canada.*

A Fibonacci-type sequence is defined by:

$$G_{n+2} = G_{n+1} + G_n ,$$

with $G_1 = a$ and $G_2 = b$. Find the minimum positive values of integers a and b, subject to a being odd, to satisfy:

$$G_{n-1}G_{n+1} - G_n^2 = -11111(-1)^n \quad \text{for } n > 1.$$

B-165 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Define the sequence $\{b(n)\}$ by

$$b(1) = b(2) = 1, \quad b(2k) = b(k), \quad \text{and} \quad b(2k+1) = b(k+1) + b(k)$$

for $k \geq 1$. For $n \geq 1$, show the following:

(a) $$b([2^{n+1} + (-1)^n]/3) = F_{n+1} .$$

(b) $$b([7 \cdot 2^{n-1} + (-1)^n]/3) = L_n .$$

SOLUTIONS

A MULTIPLICATIVE ANALOGUE

B-142 *Proposed by William D. Jackson, SUNY at Buffalo, Amherst, N.Y.*

Define a sequence as follows: $A_1 = 2$, $A_2 = 3$, and $A_n = A_{n-1}A_{n-2}$ for $n > 2$. Find an expression for A_n.

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

Let $B_n = \ln A_n$ for $n \geq 1$. Then $B_1 = \ln 2$, $B_2 = \ln 3$ and $B_n = B_{n-1} + B_{n-2}$ for $n > 2$. Clearly

$$B_n = F_{n-2} \cdot \ln 2 + F_{n-1} \cdot \ln 3$$

for $n > 2$, or

$$A_n = 2^{F_{n-2}} \cdot 3^{F_{n-1}}$$

for $n > 2$.

Also solved by Christine Anderson, Richard L. Breisch, Timothy Burns, Herta T. Freitag, J. A. H. Hunter (Canada), John Ivie, Bruce W. King, Leslie M. Klein, Arthur Marshall, C. B. A. Peck, John Wessner, Gregory Wulczyn, Michael Yoder, David Zeitlin, and the proposer.

THE DETERMINANT VANISHES

B-143 *Proposed by Raphael Finkelstein, Tempe, Arizona.*

Show that the following determinant vanishes when a and d are natural numbers:

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} .$$

What is the value of the determinant one obtains by replacing each Fibonacci number by the corresponding Lucas number?

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico

Let $r = F_{6d}/F_{3d}$ and $s = F_{6d+1} - rF_{3d+1}$. Then

$$rF_{3d} + sF_0 = F_{6d} \quad \text{and} \quad rF_{3d+1} + sF_1 = F_{6d+1}.$$

It follows by induction that

$$F_{n+6d} = rF_{n+3d} + sF_n$$

for all n; in particular, it is true for $n = a$, $n = a + d$, and $n = a + 2d$. Hence the three rows of the matrix are linearly dependent and the determinant is zero.

If each Fibonacci number is replaced by the corresponding Lucas number, the determinant will also be zero by similar reasoning.

Editorial Note: It can be shown that $r = L_{3d}$ and $s = (-1)^{d+1}$.

Also solved by F. D. Parker, C. B. A. Peck, David Zeitlin and the proposer.

LUCAS ALPHAMETIC

B-144 *Proposed by J. A. H. Hunter, Toronto, Canada.*

In this alphametic each distinct letter stands for a particular but different digit, all ten digits being represented here. It must be the Lucas series, but what is the value of the SERIES?

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    ONE
  THREE
  START
      L
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 SERIES
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Solution by Charles W. Trigg, San Diego, California

Since they are the initial digits of integers, none of θ, T, S, or L can be zero. Proceeding from the left, clearly S = 1, E = 0, and T is 8 or 9. In either event, H + T > 10, so T = 8. Then from the units' column, L = 3.

The three integer columns then establish the equalities:

$$N + R + 1 = 10$$
$$\theta + R + A + 1 = I + 10$$
$$H + 8 + 1 = R + 10 .$$

Whereupon, $N + H = 10$ and $(N,H) = (4,6)$ or $(6,4)$. But $H = R + 1$, so $R = 5$, $H = 6$, $N = 4$.

Then $\theta + A = I + 4$, and $\theta = 9$, $A = 2$, $I = 7$. (θ and A may be interchanged.) Consequently, S E R I E S = 105701.

Also solved by Richard R. Breisch, Timothy Burns, A. Gommel, Edgar Karst, John Milsom, C. B. A. Peck, John Wessner, Michael Yoder, and the proposer.

BINARY N-TUPLES

B-145 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Given an unlimited supply of each of two distinct types of objects, let f(n) be the number of permutations of n of these objects such that no three consecutive objectives are alike. Show that $f(n) = 2F_{n+1}$, where F_n is the n^{th} Fibonacci number.

Solution by Bruce W. King, Adriondack Community College, Glen Falls, N.Y.

Call a permutation of the required type an "admissible n permutation," and let A and B be two of the distinct types of objects. A list of admissible n + 1 permutations can be constructed in the following way:

(a) For each admissible n permutation ending in A, adjoin B on the right; for each distinct admissible n permutation ending in B, adjoin on the right.

(b) For each distinct admissible n - 1 permutation ending in A, adjoin BB on the right; for each distinct admissible n - 1 permutation ending in B, adjoin AA on the right.

Certainly the resulting list contains only admissible n + 1 permutations. Furthermore, there is no possibility of duplication because the permutations described in (b) end with two identical letters, but those described in (a) end with two different letters. Lastly, no n + 1 permutation is unobtainable in this way. For, if there were such a permutation, either the n - 1 permutation excluding its last two letters, or the n permutation excluding its last letter would have to be admissible. Consequently, we see that

$$f(n+1) = f(n-1) + f(n)\,.$$

The rest is an easy proof by induction.

By direct enumeration, $f(3) = 6 = 2F_4$. If $f(n) = 2F_{n+1}$ for integers $n \le N$, then

$$f(N+1) = f(N-1) + f(N) = 2F_N + 2F_{N+1} = 2(F_N + F_{N+1}) = 2F_{N+2}\,,$$

and the proof is complete.

Also solved by J. L. Brown, Jr., C. B. A. Peck, Michael Yoder, and the proposer.

ANGLES OF A TRIANGLE

B-146 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Show that $\pi = \text{Arctan}\,(1/F_{2n}) + \text{Arctan}\,F_{2n+1} + \text{Arctan}\,F_{2n+2}\,.$

Solution by C. B. A. Peck, Ordnance Research Lab., State College, Pa.

From the solution to H-82 (FQ, 6, 1, 52-54), we get

$$\text{Arctan}\,(1/F_{2n}) = \text{Arctan}\,(1/F_{2n+2}) + \text{Arctan}\,(1/F_{2n+1}).$$

The result now follows from $\text{Arctan}\,x + \text{Arctan}\,(1/x) = \pi/2$.

Also solved by Herta T. Freitag, John Ivie, Bruce W. King, John Wessner, Gregory Wulczyn, Michael Yoder, and the proposer.

TWIN PRIMES

B-147 *Proposed by Edgar Karst, University of Arizona, Tuscon, Arizona, in honor of the 66th birthday of Hansraj Gupta on Oct. 9, 1968.*

Let

$$S = (1/3 + 1/5) + (1/5 + 1/7) + \cdots + (1/32717 + 1/32719)$$

be the sum of the sum of the reciprocals of all twin primes below 2^{15}. Indicate which of the following inequalities is true:

(a) $S < \pi^2/6$ (b) $\pi^2/6 < S < \sqrt{e}$ (c) $\sqrt{e} < S$.

Solutions by Paul Sands, Student, University of New Mexico, Albuquerque, New Mexico, and the proposer. (Both used electronic computers.)

	Proposer	Paul Sands
True inequality	(b)	(b)
Number of pairs of primes involved	55	55
S, to six decimal places	1.647986	1.648627

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(Continued from p. 210.)

6. $$T_n = -(-i)^n$$

7. $$T_{n+1} = 5T_n - 6T_{n-1}$$

$$T_n = 2^n + 3^{n-1}$$

8. $$r = \frac{5 + \sqrt{29}}{2}, \qquad s = \frac{5 - \sqrt{29}}{2}$$

$$T_n = \frac{r^n - s^n}{\sqrt{29}} \text{ with terms } 1, 5, 26, 135, \cdots$$

$$V_n = r^n + s^n \text{ with terms } 5, 27, 140, \cdots$$

9. $$r = \frac{3 + i\sqrt{11}}{2}, \qquad s = \frac{3 - i\sqrt{11}}{2}$$

$$T_n = \left(\frac{33 - 16i\sqrt{11}}{55}\right)r^n + \left(\frac{33 + 16i\sqrt{11}}{55}\right)s^n$$

10. $$T_{n+1} = 5T_n + 2T_{n-1}; \quad T_1 = 3, \quad T_2 = 7.$$

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