ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-158 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

If $f_n(x)$ be the Fibonacci polynomial as defined in H-127, show that

(a) for integral values of x, $f_n(x)$ and $f_{n+1}(x)$ are prime to each other

(b) $$\left\{1 + \sum_1^n (1/f_{2n-1}f_{2n+1})\right\}\left\{1 - x^2\sum_1^n (1/f_{2n}f_{2n+2})\right\} = 1 .$$

H-159 Proposed by Clyde Bridger, Springfield College, Springfield, Illinois.

Let

$$D_k = \frac{c^k - d^k}{c - d}$$

and

$$E_k = c^k + d^k ,$$

where c and d are the roots of $z^2 = az + b$. Consider the four numbers e, f, x, y, where $e = c^k$ and $f = d^k$ are the roots of $z^2 - z E_k + (-b)^k = 0$ and y is the harmonic conjugate of x with respect to e and f. Find y when

$$x = \frac{D_{nk+k}}{D_{nk}} \qquad (k \neq 0) .$$

H-160 Proposed by D. and E. Lehmer, University of California, Berkeley, California.

Find the roots and the discriminant of

$$x^3 - (-1)^k 3x - L_{3k} = 0 .$$

H-161 Proposed by David Klarner, University of Alberta, Edmonton, Alberta, Canada.

Let

$$b(n) = \sum_{a_1+a_2+\cdots+a_i=n} \binom{a_1 + a_2}{a_2}\binom{a_2 + a_3}{a_3}\cdots\binom{a_{i-1} + a_i}{a_i} ,$$

where the sum is extended over all compositions of n and the contribution to the sum is 1 when there is only one part in the composition. Find an asymptotic estimate for $b(n)$.

SOLUTIONS

MULTI-VARIABLE SERIES

H-126 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. Sum the series

$$\sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n , \qquad \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n .$$

Sum the series

$$\sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} F_m F_n x^m y^n , \qquad \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} L_m L_n x^m y^n .$$

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} F_{n+p}\, F_{p+m}\, F_{m+n}\, x^m y^n z^p \ .$$

Solution by the Proposer.

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = \tfrac{1}{2}(1 - \sqrt{5}) \ ,$$

so that

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} \ .$$

Then

$$\begin{aligned} S_1 &= \sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n \\ &= \frac{1}{\alpha - \beta}\left\{\sum_{m,n=0}^{\infty} F_m F_n \alpha^{m+n} x^m y^n - \sum_{m,n=0}^{\infty} F_m F_n \beta^{m+n} x^m y^n\right\} \\ &= \frac{1}{\alpha - \beta}\left\{\frac{\alpha x}{1 - \alpha x - \alpha^2 x^2}\,\frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} - \frac{\beta x}{1 - \beta x - \beta^2 x^2}\,\frac{\beta y}{1 - \beta y - \beta^2 y^2}\right\} \ . \end{aligned}$$

Since $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$, it follows that

$$\begin{aligned} 1 - \alpha x - \alpha^2 x^2 &= (1 - \alpha^2 x)(1 + x) \ , \\ 1 - \beta x - \beta^2 x^2 &= (1 - \beta^2 x)(1 + x) \ , \end{aligned}$$

so that

$$S_1 = \frac{1}{(\alpha - \beta)(1 + x)(1 + y)} \left\{ \frac{\alpha x}{1 - \alpha^2 x} \frac{\alpha y}{1 - \alpha^2 y} - \frac{\beta x}{1 - \beta^2 x} \frac{\beta y}{1 - \beta^2 y} \right\}$$

$$= \frac{xy}{(\alpha - \beta)(1 + x)(1 + y)} \frac{\alpha^2[1 - \beta^2(x + y) + \beta^4 xy] - \beta^2[1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)}.$$

This reduces to

$$S_1 = \frac{xy - x^2y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

$$S_2 = \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} F_m F_n \alpha^{m+n} x^m y^n + \sum_{m,n=0}^{\infty} F_m F_n \beta^{m+n} x^m y^n$$

$$= \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2} \frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} + \frac{\beta x}{1 - \beta x - \beta^2 x^2} \frac{\beta y}{1 - \beta y - \beta^2 y^2}$$

$$= \frac{1}{(1 + x)(1 + y)} \left\{ \frac{\alpha x}{1 - \alpha^2 x} \frac{\alpha y}{1 - \alpha^2 y} + \frac{\beta x}{1 - \beta^2 x} \frac{\beta y}{1 - \beta^2 y} \right\}$$

$$= \frac{xy}{(1 + x)(1 + y)} \frac{\alpha^2[1 - \beta^2(x + y) + \beta^4 xy] + \beta^2[1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)}$$

$$= \frac{xy}{(1 + x)(1 + y)} \frac{L_2 - 2(x + y) + L_2 xy}{(1 - 3x + x^2)(1 - 3y + y^2)}$$

and therefore

$$S_2 = \frac{3xy - 2xy(x + y) + 3x^2y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Clearly

$$\sum_{\substack{m,n=0\\ m+n\text{ even}}}^{\infty} F_mF_nx^my^n = \frac{1}{2}\left\{\sum_{m,n=0}^{\infty} F_mF_nx^my^n + \sum_{m,n=0}^{\infty}(-1)^{m+n}F_mF_nx^my^n\right\}$$

$$= \frac{1}{2}\left\{\frac{xy}{(1-x-x^2)(1-y-y^2)} + \frac{xy}{(1+x-x^2)(1+y-y^2)}\right\}$$

$$= \frac{xy}{2}\,\frac{(1+x-x^2)(1+y-y^2)+(1-x-x^2)(1-y-y^2)}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}$$

$$= \frac{xy-(x^2+y^2)xy+x^2y^2+x^3y^3}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)} .$$

Similarly,

$$\sum_{\substack{m,n=0\\ m+n\text{ even}}}^{\infty} L_mL_nx^my^n = \frac{1}{2}\left\{\sum_{m,n=0}^{\infty} L_mL_nx^my^n + \sum_{m,n=0}^{\infty}(-1)^{m+n}L_mL_nx^my^n\right\}$$

$$= \frac{1}{2}\quad\frac{(2-x)(2-y)}{(1-x-x^2)(1-y-y^2)} + \frac{(2+x)(2+y)}{(1+x-x^2)(1+y-y^2)}$$

$$= \frac{1}{2}\,\frac{(2-x)(2-y)(1+x-x^2)(1+y-y^2)+(2+x)(2+y)(1-x-x^2)(1-y-y^2)}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)}$$

$$= \frac{4+xy-6(x^2+y^2)+5x^2y^2+x^3y^3}{(1-3x^2+x^4)\cdot(1-3y^2+y^4)} .$$

Now for the last series, we have

$$S = \frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=o}^{\infty}(\alpha^{n+p}-\beta^{n+p})(\alpha^{p+m}-\beta^{p+m})(\alpha^{m+n}-\beta^{m+n})x^my^nz^p$$

$$= \frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=o}^{\infty}(\alpha^{2m+2n+2p}-\beta^{2m+2n+2p})x^my^nz^p$$

$$-\frac{1}{(\alpha-\beta)^3}\sum_{m,n,p=0}^{\infty}\sum_{m,n,p}(\alpha^{2m+n+p}\beta^{n+p}-\alpha^{n+p}\beta^{2m+n+p})x^my^nz^p$$

$$= \frac{1}{(\alpha-\beta)^3}\left\{\frac{1}{(1-\alpha^2x)(1-\alpha^2y)(1-\alpha^2z)} - \frac{1}{(1-\beta^2x)(1-\beta^2y)(1-\beta^2z)}\right\}$$

$$-\frac{1}{(\alpha-\beta)^3}\left\{\sum_{x,y,z}\frac{1}{(1-\alpha^2x)(1+y)(1+z)} - \sum_{x,y,z}\frac{1}{(1-\beta^2x)(1+y)(1+z)}\right\}$$

$$= \frac{1}{5}\,\frac{\Sigma x - 3\Sigma xy + 8xyz}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)} - \frac{1}{5}\sum_{x,y,z}\frac{x}{(1-3x+x^2)(1+y)(1+z)} .$$

The sums

$$\sum_{m,n,p}, \sum_{x,y,z}$$

indicate summation over all permutations of the letters indicated.

We find, after some computation, that

$$S = \frac{b - 5c - 2ac + 2bc - b^2 + c^2}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1-3y+y^2)(1-3z+z^2)},$$

where

$$a = x + y + z, \quad b = yz + zx + xy, \quad c = xyz .$$

For $z = 0$ the above result reduces to

$$\sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n = \frac{xy - x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)}$$

in agreement with the first result, where $0^0 = 1$, by convention.

Also solved by A. Shannon, Boroko, T. P. N. G.

Note: Due to an editorial error, problem H-120 was also listed as H-127.

MOD SQUAD

H-128 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Show that

$$F_n \equiv 2^{2n+3} - 2^{3n+3} \pmod{11},$$
$$L_n \equiv 2^{2n} + 2^{3n} \pmod{11} .$$

Generalize.

Solution by the Proposer.

Let p be any prime $\equiv \pm 1 \pmod{10}$. Then it is known (Dmitri Thoro, "An Application of Unimodular Transformations," Fibonacci Quarterly, 2(1964), 291-295) that 5 is a quadratic residue (mod p), so let x_0 be a solution of $x^2 \equiv 5 \pmod{p}$. Since x_0 and $p - x_0$ are both solutions of this, one of which is odd, we may assume x_0 is odd, say $x_0 = 2a - 1$. Then

$$2a - 1 \equiv \sqrt{5}, \qquad a \equiv (1 + \sqrt{5})/2 ,$$

so that $a^2 - a - 1 \equiv 0 \pmod{p}$. Hence $x - a$ divides $x^2 - x - 1 \pmod{p}$, showing that

$$x^2 - x - 1 \equiv (x - a)(x - b) \pmod{p}$$

for some integer b. It follows that $u_n = c_1 a^n + c_2 b^n$ obeys

$$u_{n+2} \equiv u_{n+1} + u_n \pmod{p} ,$$

where c_1 and c_2 are arbitrary constants.

We first evaluate c_1 and c_2 when $u_n \equiv F_n \pmod{p}$. When $n = 0,1$, we find

$$c_1 + c_2 \equiv 0 \pmod{p}$$

$$c_1 a + c_2 b \equiv 1 \pmod{p},$$

which has a solution if and only if $a \not\equiv b \pmod{p}$, which is clearly the case here. We see that then $c_1 = 1/(a - b)$, $c_2 = -1/(a - b)$, so

$$F_n \equiv \frac{a^n - b^n}{a - b} \pmod{p}.$$

Similarly,

$$L_n \equiv a^n + b^n \pmod{p}.$$

These may be considered the Binet forms for the Fibonacci and Lucas numbers in the integers modulo p.

The above problem follows from this by noting

$$x^2 - x - 1 \equiv (x - 4)(x - 8) \pmod{11},$$

and that $1/(4 - 8) \equiv 8 \pmod{11}$.

Also solved by L. Carlitz, Duke University, and A. Shannon, Boroko, T. P. N. G.

RADICAL TSCHEBYSHEV

H-129 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define the Fibonacci polynomials by $f_1(x) = 1$, $f_2(x) = x$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, $n > 0$. Solve the equation

$$(x^2 + 4)f_n^2(x) = 4k(-1)^{n-1}$$

in terms of radicals, where k is a constant.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

It is stated in The American Mathematical Monthly, 1968, p. 295, that

$$i^{n-1} f_n(x) = U_{n-1}\left(\frac{1}{2}\, ix\right),$$

where

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$

Thus

$$\begin{aligned}(-1)^{n-1}f_n^2(x) &= U_{n-1}^2\left(\tfrac{1}{2}\, ix\right) = \sin^2\left(n \cos^{-1}\tfrac{1}{2}\, ix\right)/\sin^2\left(\cos^{-1}\tfrac{1}{2}\, ix\right) \\ &= 4\sin^2\left(n\cos^{-1}\tfrac{1}{2}\, ix\right)/(4 + x^2)\end{aligned}$$

Thus

$$(x^2 + 4)f_n^2(x) = (-1)^{n-1} 4 \sin^2(n \cos^{-1} \tfrac{1}{2} ix) .$$

By comparison,

$$k = \sin^2 (n \cos^{-1} \tfrac{1}{2} ix)$$

whence

$$x = -2i \cos\left(\frac{1}{n} \sin^{-1} \sqrt{k} \right) .$$

Note: The proposer obtained the solution,

$$x = i \quad (1 - 2k + 2\sqrt{k^2 - k})^{\frac{1}{2}n} + (1 - 2k - 2\sqrt{k^2 - k})^{\frac{1}{2}n} ,$$

where any $(2n)^{th}$ root may be taken in the first radical and the $(2n)^{th}$ root of the second radical is chosen so that their product is unity.

Also solved by A. Shannon, Boroko, T. P. N. G.

GAUCHE PASCAL

H-131 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the left-adjusted Pascal triangle. Denote the left-most column of ones as the zeroth column. If we take sums along the rising diagonals, we get Fibonacci numbers. Multiply each column by its column number and again take sums, C_n, along these same diagonals. Show $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j} F_j$$

Solution by L. Carlitz, Duke University.

Clearly,

$$C_n = \sum_{2j \le n} j \binom{n-j}{j}, \quad C_0 = C_1 = 0 .$$

Hence

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} x^n \sum_{2j \le n} j\binom{n-j}{j}$$

$$= \sum_{j=0}^{\infty} j\, x^{2j} \sum_{n=0}^{\infty} \binom{n+j}{j} x^n$$

$$= \sum_{j=0}^{\infty} j\, x^{2j} (1-x)^{-j-1}$$

$$= \frac{x^2}{(1-x)^2} \sum_{j=0}^{\infty} (j+1)x^{2j}(1-x)^{-j}$$

$$= \frac{x^2}{(1-x)^2}\left(1 - \frac{x^2}{1-x}\right)^{-2}$$

$$= x^2(1 - x - x^2)^{-2} \quad .$$

Since

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad ,$$

it follows at once that

$$C_n = \sum_{j=0}^{n} F_j \, F_{n-j} \quad .$$

Also solved by D. Zeitlin, Minneapolis, Minnesota; A. Shannon, Boroko, T. P. N. G.; and E. Frankel.

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