MULTIPLE FIBONACCI SUMS

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I. INTRODUCTION

Let us define the Fibonacci numbers by means of the recurrence relation

$$(1) \qquad F_{n+2} = F_{n+1} + F_n \quad \text{with} \quad F_1 = 1,\ F_2 = 1$$

To derive a formula for the sum of the first m Fibonacci numbers, write (1) as $F_n = F_{n+2} - F_{n+1}$, and let $n = 1, 2, 3, \cdots, m$, as shown below.

$$\begin{aligned} F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ &\vdots \\ F_{m-1} &= F_{m+1} - F_m \end{aligned} \quad .$$

Adding, we have

$$(2) \qquad \sum_{k=1}^{m} F_k = F_{m+2} - 1 ,$$

a well-known and useful result. In this paper, we shall be concerned with a generalization of (2) and its subsequent derivation, as well as another possible result.

II. DERIVATION OF FORMULA

Without stating in exact form the generalization which we shall consider, let us proceed inductively. Summing both sides of (2), we obtain

$$\sum_{m=1}^{p} \sum_{k=1}^{m} F_k = \sum_{m=1}^{p} (F_{m+2} - 1) = \sum_{m=1}^{p} F_{m+2} - \sum_{m=1}^{p} 1 = F_{p+4} - F_4 - p ,$$

as is easily seen. Summing this again,

$$\sum_{p=1}^{q} \sum_{m=1}^{p} \sum_{k=1}^{m} F_k = \sum_{p=1}^{q} (F_{p+4} - F_4 - p) = \sum_{p=1}^{q} F_{p+4} - \sum_{p=1}^{q} F_4 - \sum_{p=1}^{q} p.$$

To evaluate this, we use the well-known formula

$$(1 + 2 + \cdots + q) = \tfrac{1}{2}q(q + 1) ,$$

the sum of the first q natural numbers, to give

$$\sum_{p=1}^{q} \sum_{m=1}^{p} \sum_{k=1}^{m} F_k = F_{q+6} - F_6 - qF_4 - \frac{q(q + 1)}{2} .$$

If we sum this result again, we have

$$\sum_{q=1}^{r} \sum_{p=1}^{q} \sum_{m=1}^{p} \sum_{k=1}^{m} F_k = \sum_{q=1}^{r} \left(F_{q+6} - F_6 - qF_4 - \frac{q(q + 1)}{2} \right)$$

$$= \sum_{q=1}^{r} F_{q+6} - \sum_{q=1}^{r} F_6 - \sum_{q=1}^{r} qF_4 - \sum_{q=1}^{r} \frac{q(q + 1)}{2}$$

To evaluate

$$\frac{1}{2} \sum_{q=1}^{r} q(q + 1) ,$$

we use the fact that the sum of the first r triangular numbers is the r^{th} tetrahedral number, giving

$$\sum_{q=1}^{r} \sum_{p=1}^{q} \sum_{m=1}^{p} \sum_{k=1}^{m} F_k = F_{r+8} - F_8 - rF_6 - \frac{r(r+1)}{2} F_4 - \frac{r(r+1)(r+2)}{3!} \, .$$

Let us now generalize this procedure to the case of n summations. Thus, we consider sums of the form

$$\sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_1=0}^{a_2} \sum_{a_0=0}^{a_1} F_{a_0} \quad ,$$

where the limits in the summation are members of the sequence of arbitrary constants,

$$\left\{a_j\right\}_{j=0}^{n} \quad .$$

Examining the specific cases we have worked out, we see that the first term of our general result will be of the form F_{a_n+2n}, the second of the form F_{2n}. The third will be $a_n F_{2n-2}$, and the fourth

$$F_{2n-4}\, a_n (a_n + 1)/2 = F_{2n-4} \sum a_{n-1} \, .$$

In general, we need to evaluate sums of the form

$$\sum \cdots \sum \sum a_0 \quad .$$

To do this, we have the following result [1].

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} = f_r^{a_n} = \binom{a_n + r - 1}{r},$$

where $f_r^{a_n}$ is the r^{th} figurate number of order a_n, and r is the number of summations plus one. Thus, we conjecture that

(3)
$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} F_{a_0} = F_{a_n+2n} - \sum_{r=0}^{n-1} F_{2(n-r)} \binom{a_n + r - 1}{r}$$

III. PROOF OF FORMULA

Let us now prove our conjecture (3) by induction on n. By the principle of mathematical induction, we first check for $n = 1$, which is obviously formula (2), and is thus true. We then assume (3) is true for $n = s$, and show that $n = s + 1$ is also true. Thus, we have to show

(4)
$$\sum_{a_s=1}^{a_{s+1}} \left(F_{a_s+2s} - \sum_{r=0}^{s-1} F_{2(s-r)} \binom{a_s + r - 1}{r} \right) = F_{a_{s+1}+2(s+1)} - \sum_{r=0}^{s} F_{2(s+1-r)} \binom{a_{s+1} + r - 1}{r} .$$

To find the first summation on the left-hand side, we can easily derive

(5)
$$\sum_{a_s=1}^{a_{s+1}} F_{a_s+2s} = F_{a_{s+1}+2(s+1)} - F_{2+2s} .$$

To find the second summation, consider

$$\sum_{a_s=1}^{a_{s+1}} \sum_{r=0}^{s-1} F_{2(s-r)} \binom{a_s + r - 1}{r} = \sum_{a_s=1}^{a_{s+1}} F_{2s} \binom{a_s - 1}{0} + \cdots + F_2 \binom{a_s + s - 1 - 1}{s - 1}$$

(6)
$$= F_{2s} \sum_{a_s=1}^{a_{s+1}} \binom{a_s - 1}{0} + F_{2(s-1)} \sum_{a_s=1}^{a_{s+1}} \binom{a_s}{1} + \cdots + F_2 \sum_{a_s=1}^{a_{s+1}} \binom{a_s + s - 1 - 1}{s - 1} .$$

It can easily be established by induction that for $n \geq r$,

(7)
$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} .$$

Thus, applying (7) to (6),

$$(8)\qquad \sum_{a_{s}=1}^{a_{s+1}} \sum_{r=0}^{s-1} F_{2(s-r)}\binom{a_s + r - 1}{r} = \sum_{r=1}^{s} F_{2(s-r+1)}\binom{a_{s+1} + r - 1}{r}$$

Substituting (5) and (8) into (4), we obtain

$$F_{a_{s+1}+2(s+1)} - F_{2+2s} - \sum_{r=1}^{s} F_{2(s-r+1)}\binom{a_{s+1}+r-1}{r} = F_{a_{s+1}+2(s+1)} - \sum_{r=0}^{s} F_{2(s-r+1)}\binom{a_{s+1}+r-1}{r}$$

which proves our proposed formula.

We remark that this general formula is true for all recurrence relations of the form

$$f_{n+2} = f_{n+1} + f_n, \quad f_1 = a, \quad f_2 = b ,$$

where a and b are arbitrary integers. Thus,

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} f_{a_0} = f_{a_n+2n} - \sum_{r=0}^{n-1} f_{2(n-r)}\binom{a_n + r - 1}{r} .$$

In particular, this result is true for the Lucas numbers defined by

$$L_{n+2} = L_{n+1} + L_n, \; L_1 = 1, \; L_2 = 3 .$$

IV. OTHER RESULTS

We shall develop a formula similar to (3), but which is derived by a different method and gives rise to a new identity. To use this method, we need a result of Hoggatt [2], namely that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad \frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty}\left(\sum^{n}\sum \cdots \sum a_j\right)x^n ,$$

where there are m summations in the coefficient of x^n. Thus, the multiple sums are the convolutions of the a_j's with the elements of the m^{th} column of Pascal's left-adjusted triangle. Letting

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = x(1 - x - x^2)^{-1} ,$$

then

$$\frac{f(x)}{(1-x)^m} = \frac{x}{1-x-x^2}(1-x)^{-m} = \frac{x}{1-x-x^2}\sum_{j=0}^{\infty}(-1)^j\binom{-m}{j}x^j$$

$$= \frac{1}{1-x-x^2}\sum_{j=0}^{\infty}\binom{m+j-1}{j}x^{j+1} .$$

If we carry out the indicated long division, then

$$\frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty}\left(\sum^{n}\cdots\sum F_j\right)x^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} F_{n-j}\binom{m+j-1}{j}\right)x^n .$$

Equating coefficients of x^n, and using the notation of (3), we get

$$\text{(9)} \qquad \sum_{a_{n-1}=0}^{a_n}\ \sum_{a_{n-2}=0}^{a_{n-2}} \cdots \sum_{a_0=0}^{a_1} F_{a_0} = \sum_{j=0}^{a_n} F_{a_n-j}\binom{n+j-1}{j} .$$

By equating (3) and (9), we derive the following identity

$$\text{(10)} \qquad \sum_{j=0}^{a_n} F_{a_n-j}\binom{n+j-1}{j} = F_{a_n+2n} - \sum_{j=0}^{n-1} F_{2(n-j)}\binom{a_n+j-1}{j} .$$

We now note that this method can be used to find a general formula for

$$\sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-1}=0}^{a_{n-1}} \cdots \sum_{a_0=0}^{a_1} b_{a_0} ,$$

where $\left\{b_j\right\}_0^\infty$ is a sequence of real numbers. Since

$$f(x) = \sum_{n=0}^{\infty} b_n x^n ,$$

then

$$\frac{f(x)}{(1-x)^m} = \sum_{n=0}^{\infty} b_n x^n \cdot (1-x)^{-m} = \sum_{n=0}^{\infty} b_n x^n \cdot \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b_{n-j} \binom{m+j-1}{j} \right) x^n ,$$

by definition of the Cauchy product of two infinite series. Thus,

$$\sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-2}=0}^{a_{n-1}} \cdots \sum_{a_0=0}^{a_1} b_{a_0} = \sum_{j=0}^{a_n} b_{a_n-j} \binom{n+j-1}{j} .$$

This then gives a generalization of (10) for recurrence relations of the form

$$f_{n+2} = f_{n+1} + f_n , \quad f_1 = a, \quad f_2 = b ,$$

where a and b are arbitrary integers, namely

$$\sum_{j=0}^{a_n} f_{a_n-j} \binom{n+j-1}{j} = f_{a_n+2n} - \sum_{j=0}^{n-1} f_{2(n-j)} \binom{a_n+j-1}{j} . \tag{11}$$

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REFERENCES

1. E. Dickson, History of the Theory of Numbers, Vol. 2, p. 7.
2. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, p. 228.

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