EXPLICIT DETERMINATION OF THE PERRON MATRICES IN PERIODIC ALGORITHMS OF THE PERRON-JACOBI TYPE WITH APPLICATION TO GENERALIZED FIBONACCI NUMBERS WITH TIME IMPULSES

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0. By the Perron matrices P_k in an n-dimensional algorithm of the Jacobi-Perron type [1] we understand the analogue to the 2-dimensional matrices

$$\begin{bmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{bmatrix}$$

built up from two consecutive "convergents"

$$\frac{p_{k-1}}{q_{k-1}}, \quad \frac{p_k}{q_k} \quad .$$

of an ordinary continued fraction.

As explained in detail in Chapter I of a previous joint paper of ours [2] these $n \times n$ matrices P are defined recurrently by

$$P_k = P_{k-1}A \qquad (k = 0, 1, \cdots) ,$$

with the initial condition

$$P_{-1} = I \qquad \text{(n-rowed unit matrix)} ,$$

where the matrices

$$A_k = \begin{bmatrix} 0 & \cdots & 0 & a_{0k} \\ 1 & & & a_{1k} \\ & \ddots & & \\ & & 1 & a_{n-1,k} \end{bmatrix} \qquad (k = 0, 1, \cdots) ,$$

are built up from the "partial quotients"

$$a_{0k} = 1, \quad a_{1k}, \cdots a_{n-1},$$

in the algorithm, which in the special case $n = 2$ of ordinary continued fractions reduce essentially to only one a_{1k} in each step.

From this recurrent definition it follows that the Perron matrices P_{-1} are built up from an infinite sequence of n-termed columns $\mathfrak{M}_{k-1}$ in the form

$$P_{k-1} = (\mathfrak{M}_{k-n}, \cdots, \mathfrak{M}_{k-1}) ,$$

satisfying the recurrency formulae

$$(0.1) \qquad a_{0k}\mathfrak{M}_{k-n} + \cdots + a_{n-1,k}\mathfrak{M}_{k-1} \qquad (k \geq 0) ,$$

with the initial condition that

$$\mathfrak{M}_{-n} = W_0, \cdots, \mathfrak{M}_{-1} = W_{n-1}$$

are the columns of the n-rowed unit matrix I.

In the present paper the entries of the Perron matrices P_{k-1} shall be denoted by $p^{(\nu)}_{k-(n-\nu')}$, where the super- and subscripts $\nu = 0, \cdots, n-1$ and $\nu' = 0, \cdots, n-1$ indicate the lines and columns, respectively:

$$P_{k-1} = (p^{(\nu)}_{k-(n-\nu')})_{\substack{\nu \text{ lines} \\ \nu' \text{ columns}}} \qquad \begin{pmatrix} \nu = 0, \cdots, n-1 \\ \nu' = 0, \cdots, n-1 \end{pmatrix}$$

Thus the recurrency formulae (0.1) with the initial conditions (0.2) become

$$(0.3) \qquad p^{(\nu)}_k = \sum_{\nu'=0}^{n-1} a_{\nu'} \, p^{(\nu)}_{k-(n-\nu')} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix}$$

with

$$(0.4) \qquad p^{(\nu)}_{-(n-\nu')} = e^{(\nu)}_{\nu'} = \begin{Bmatrix} 1 \text{ for } \nu = \nu' \\ 0 \text{ for } \nu \neq \nu' \end{Bmatrix}$$

(entries of the unit matrix I) .

In Perron's original paper [2] these $p^{(\nu)}$ would be the $A^{(k+n)}$.

We shall consider only purely periodical algorithms. Let ℓ be the length of the period. Then in the recurrency formulae (0.3) there are only ℓ different n-termed coefficient sets $a_{\nu' k}$ $(\nu' = 0,\cdots,n-1)$, which recur periodically. In our first, purely algebraic part these ℓ sets will be considered as algebraically independent indeterminates and denoted by $a_{\nu'}^{(\lambda)}$ $(\lambda = 0,\cdots,\ell-1)$. For the sake of algebraic generality and formal symmetry we include in this stipulation also the coefficients $a_0^{(\lambda)}$ which in the actual algorithm are throughout equal to 1.

For purely periodical algorithms, the infinite sequence of recurrency formulae (0.3) reduces to a finite system

$$p_{k\ell+\lambda}^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')} \qquad \begin{pmatrix} k & 0 \\ \lambda=0,\cdots,\ell-1 \\ \nu=0,\cdots,n-1 \end{pmatrix} \tag{0.5}$$

of ℓ linear recurrencies with the n linearly independent initial conditions (0.4).

We shall chiefly be concerned with the special case of period length $\ell = 1$, where there remains only one single linear recurrency

$$p_k^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{pmatrix} \tag{0.6}$$

with the n linearly independent initial conditions (0.4). In this case we shall obtain the following simple explicit expressions for the entries $p_k^{(\nu)}$ of the Perron matrices P_k (last column):

$$(0.7)\quad \left\{ \begin{aligned} p_k^{(\nu)} = & \sum_{\substack{L(k_0,\cdots,k_{n-1})=k+(n-\nu) \\ k_0,\cdots,k_{n-1} \geq 0}} \begin{pmatrix} k_0+\cdots+k_{n-1} \\ k_0,\cdots,k_{n-1} \end{pmatrix} \\ & \frac{k_0+\cdots+k_\nu}{k_0+\cdots+k_{n-1}}\, a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \begin{pmatrix} k \geq 0 \\ \nu=0,\cdots,n-1 \end{pmatrix} \end{aligned} \right. ,$$

with summation restricted by the linear form

$$L(k_0, \cdots, k_{n-1}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1} \tag{0.8}$$

in the summation variables $k_0, \cdots, k_{n-1}$, and with the polynomial coefficients

$$\binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots \; k_{n-1}} = \frac{(k_0 + \cdots + k_{n-1})!}{k_0! \cdots k_{n-1}!} \quad . \tag{0.9}$$

The procedure by which we reach our aim (0.7) is the very old method of Euler, viz., to translate the recurrency formula (0.6) for the sequences $p_k^{(\nu)}$ into algebraic expressions for the generating functions

$$p^{(\nu)}(x) = \sum_{k \geq 0} p_k^{(\nu)} x^k \qquad (\nu = 0, \cdots, n-1) \; ,$$

and to determine the power series coefficients $p_k^{(\nu)}$ from those algebraic expressions.

In the general case of arbitrary period length ℓ we shall show that the same object can be achieved in principle. The explicit formulae, however, would be so complicated that one can hardly expect to write them down in extenso, but for simpler special cases. As an example, we shall carry through in extenso the very special case $\ell = 2$ with $n = 2$, i.e., the case of purely periodic ordinary continued fractions with period length 2.

There is, however, a special case of a more general type in which we can obtain as definite a result as (0.7). Amongst the numerous periodic algorithms, discovered by the first author in previous papers*, a particular period structure prevails, viz., of length $\ell = n$ and with the following specialization of the coefficients in (0.5):

$$a_{\nu'}^{(\lambda)} = t^{d_n(\lambda, \nu')} a_{\nu'} \qquad (\lambda, \nu' = 0, \cdots, n-1) \; , \tag{0.10}$$

where

*See the complete list of references in [3].

$$d_n(\lambda, \nu') = \begin{matrix} 0 \text{ for } \lambda + \nu' < n \\ 1 \text{ for } \lambda + \nu' \geq n \end{matrix} \tag{0.11}$$

is the so-called "number to be carried over" in the addition of the n-adic digits λ, ν'. In this important case we shall derive from (0.7) the following generalization:

$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{L(k_0,\cdots,k_{n-1})=k+(n-\nu)} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} \times$$
$$t^{k_0+\cdots+k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \quad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right) . \tag{0.12}$$

We shall come back to another significance of this case in our second chapter.

CHAPTER I: ALGEBRAIC FOUNDATIONS

1. We begin with considering the special case of period length $\ell = 1$. To the recurrency formula (0.6), viz.,

$$p^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p^{(\nu)}_{-(n-\nu')} \qquad (k \geq 0) \tag{1.1}$$

with the initial conditions (0.4), viz.,

$$p^{(\nu)}_{-(n-\nu')} = e^{(\nu)}_{\nu'} \qquad (\nu, \nu' = 0, \cdots, n-1) \quad , \tag{1.2}$$

we let correspond the characteristic polynomial

$$F = F(x) = 1 - \sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'} \quad ,$$

and the n generating functions

$$p^{(\nu)} = p^{(\nu)}(x) = \sum_{k \geq 0} p_k^{(\nu)} x^k .$$

Now

$$\begin{aligned}
a_{\nu'} x^{n-\nu'} p^{(\nu)} &= \sum_{k \geq 0} a_{\nu'} p_k^{(\nu)} x^{k+(n-\nu')} \\
&= \sum_{k \geq n-\nu'} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&= \sum_{k \geq 0} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&\quad - \sum_{0 \leq k \leq (n-\nu')-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&= \sum_{k \geq 0} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} x^k \\
&\quad - \left\{ \begin{matrix} a_{\nu'} x^{\nu-\nu'} & \text{for } \nu' \leq \nu \\ 0 & \text{for } \nu' > \nu \end{matrix} \right\} ,
\end{aligned}$$

the latter because the summation condition $0 \leq k \leq (n - \nu') - 1$ is equivalent to $-(n - \nu') < k - (n - \nu') \leq -1$, so that the initial conditions (1.2) are applicable.

Summation over ν' then yields

$$\begin{aligned}
FP^{(\nu)} - P^{(\nu)} &= -\sum_{k \geq 0} \left(\sum_{\nu'=0}^{n-1} a_{\nu'} p_{k-(n-\nu')}^{(\nu)} \right) x^k + \\
&\quad + \sum_{\nu'=0}^{\nu} a_{\nu'} x^{\nu-\nu'} .
\end{aligned}$$

Here the negative terms on the left and right are equal to each other on account of the recurrency formula (1.1). This gives the algebraic expressions

$$p^{(\nu)} = \frac{A^{(\nu)}}{F} \quad \text{with} \quad A^{(\nu)}$$

(1.3)
$$= A^{(\nu)}(x) = \sum_{\nu'=0}^{\nu} a_{\nu'} x^{\nu-\nu'} \qquad (\nu = 0,\cdots,n-1) ,$$

for the generating functions $P^{(\nu)}$.

2. In order to obtain explicit expressions for the recurrent sequences $p_k^{(\nu)}$, we have to develop the rational functions (1.3) into power series in x. The power series for $1/F$ is obtained easily from the geometrical series:

(2.1)
$$\begin{aligned}\frac{1}{F} &= \sum_{k\geq 0}\left(\sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'}\right)^k \\ &= \sum_{k_0,\cdots,k_{n-1}\geq 0} \binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}} \times \\ &\quad \times a_0^{k_0}\cdots a_{n-1}^{k_{n-1}} x^{nk_0+(n-1)k_1+\cdots+1k_{n-1}} \\ &= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}} a_0^{k_0}\cdots a_{n-1}^{k_{n-1}}\right)x^k ,\end{aligned}$$

with the linear form

(2.2)
$$L(\mathfrak{M}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1}$$

in the summation variable vector

$$\mathfrak{M} = (k_0, \cdots, k_{n-1}) .$$

In what follows the summation variables $k_0,\cdots,k_{n-1}$ are throughout silently supposed to be 0. The solutions $\mathfrak{M}$ of $L(\mathfrak{M}) = k$ correspond to the partitions of k into summands from $1,\cdots,n$; their number $p_{n(k)}$ is well known.

In order to obtain from (2.1) the power series for the rational functions $p^{(\nu)}$ in (1.3), we have to multiply by the single terms $a_{\nu'}x^{\nu-\nu'}$ of the polynomials $A^{(\nu)}$ in the numerator and then sum up over ν'. Multiplication by one of these terms and subsequent transformation of the summation yields in the first place

$$\begin{aligned}\frac{a_{\nu'}x^{\nu-\nu'}}{F} &= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right.\\ &\qquad \left.\times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^{k+(\nu-\nu')}\\ &= \sum_{k\geq 0}\left(\sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right.\\ &\qquad \left. a_0^{k_0}\cdots a_{'}^{k_{'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^{k}\ .\end{aligned}$$

In order to simplify the subsequent summation over ν' we have here formally admitted terms with $L(k_0,\cdots,k_{n-1}) < 0$, which actually vanish because the summation condition is empty. Summation over ν' then yields the development

$$\begin{aligned}p^{(\nu)} &= \sum_{k\geq 0}\left(\sum_{\nu'=0}^{\nu}\ \sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\right.\\ &\qquad \left.\times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}}\right)x^{k}\end{aligned}$$

for the generating functions, and thus the explicit expressions

$$\begin{aligned}p^{(\nu)} &= \sum_{\nu'=0}^{\nu}\ \sum_{L(\mathfrak{M})=k-(\nu-\nu')}\binom{k_0+\cdots+k_{n-1}}{k_0,\ \cdots,\ k_{n-1}}\\ &\qquad\qquad \times\, a_0^{k_0}\cdots a_{\nu'}^{k_{\nu'}+1}\cdots a_{n-1}^{k_{n-1}}\ ,\end{aligned} \tag{2.3}$$

for the recurrent sequences in question.

3. As a last step, the sum (2.3) of polynomials in $a_0, \cdots, a_{n-1}$ can be put into canonical form, i.e., represented as a single polynomial in $a_0, \cdots, a_{n-1}$. This is achieved by a further transformation of summation which, in its turn, allows to reverse the order of the two summations.

The transformation, leading to this, is

$$k_{\nu'} \to k_{\nu'} - 1 \ . \tag{3.1}$$

It is true that by it the silent summation condition $k_{\nu'} \geq 0$ is transformed into $k_{\nu'} \geq 1$. However here, too, after the transformation, the summation may again be extended formally over all $k_{\nu'} \geq 0$, because the polynomial coefficients with a negative term in the "denominator" vanish, if only the sum of all terms in the "numerator" is non-negative. The truth of this assertion is easily seen by expressing the factorials in the definition (0.9) of the polynomial coefficients as values of the Gamma-function and observing that this function has no zeros at all, and has poles only at $0, -1, -2, \cdots$. That the "numerator" here is non-negative, is seen as follows. Under the transformation (3.1), according to the definition (0.8), one has

$$L(\mathfrak{M}) \to L(\mathfrak{M}) - (n - \nu')$$

and hence

$$p_\kappa^{(\nu)} = \sum_{\nu'=0}^{\nu} \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0, \ \cdots, \ k_{\nu'} - 1, \ \cdots, \ k_{n-1}} \times a_0^{k_0} \cdots a_{\nu'}^{k_{\nu'}} \cdots a_{n-1}^{k_{n-1}} \ . \tag{3.2}$$

Here the sum of all terms in the "numerator" is surely non-negative, because $L(\mathfrak{M}) = k + (n - \nu) \geq k + 1 \geq 1$ and hence not all $k_0, \cdots, k_{n-1}$ vanish.

Since by this transformation the inner summation condition in (3.2) has become independent of the outer summation variable ν', the order of the two summations may now be reversed:

$$p_k^{(\nu)} = \sum_{L(\mathfrak{M})=k+(n-\nu)} \left(\sum_{\nu'=0} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots,\ k_{n-1}} \right) \times \tag{3.3}$$
$$\times\, a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \ .$$

Thus the polynomial (2.3) has already been put into canonical form. But, moreover, it is even possible to consummate the inner sum in (3.3). For, by definition

$$\binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots,\ k_{n-1}} = \frac{(k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1})!}{k_0! \cdots (k_{\nu'} - 1)! \cdots k_{n-1}!}$$
$$= \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \frac{k_{\nu'}}{k_0 + \cdots + k_{n-1}} \qquad (\text{also for } k_{\nu'} = 0) \ .$$

and hence

$$\sum_{\nu'=0}^{\nu} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{\nu'} - 1\ ,\ \cdots\ ,\ k_{n-1}} \tag{3.4}$$
$$= \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} \ .$$

Thus (3.3) yields our first chief result

$$p_k^{(\nu)} = \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + k_{n-1}}{k_0\ ,\ \cdots\ ,\ k_{n-1}} \times \tag{3.5}$$
$$\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \ , \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix} \right) \ .$$

as announced in (0.7).

We remark that (3.5), conveniently interpreted, holds even for $k \geq -n$, i.e., including the initial values corresponding to $k = -(n - \nu')$ $(\nu' = 0, \cdots, n - 1)$. For in these cases the summation condition $L(\mathfrak{M}) = \nu' - \nu$ has no non-negative solutions if $\nu' < \nu$, only one such solution, viz., $k_0, \cdots, k_{n-1} = 0$, if $\nu' = \nu$, and only such solutions with $k_0, \cdots, k_\nu = 0$ if $\nu' > \nu$. Hence for $\nu' < \nu$ the sum is 0 by the usual convention for empty sums, for $\nu' > \nu$ it is also zero with regard to the factor

$$\frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} ,$$

and for $\nu' = \nu$ it is 1 if this factor of the indeterminate form $0/0$ is understood as 1.

It is furthermore perhaps not useless to remark that for the first initial condition (1.2), i.e., for $\nu = 0$ this result can also be written in the simpler form

$$(3.6) \qquad p_k^{(0)} = \sum_{L(\mathfrak{M})=k} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} a_0^{k_0+1} \cdots a_1^{k_1} \cdots a_{n-1}^{k_{n-1}} \qquad (k \geq 0)$$

as is already clear from the intermediate result (2.3).

4. Since operating with polynomial coefficients, and in particular with their fundamental recurrency property

$$(4.1) \qquad \sum_{\nu'=0}^{n-1} \binom{k_0 + \cdots + (k_{\nu'} - 1) + \cdots + k_{n-1}}{k_0, \cdots, k_{\nu'} - 1, \cdots, k_{n-1}} = \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}}$$

(special case $\nu = n - 1$ of (3.4)), is not so familiar and handy as in the special case $n = 2$ of binomial coefficients, we attach here the following simple reduction of the former to the latter.

From the definition (0.9) one has

$$(4.2) \qquad \begin{aligned} \binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}} &= \binom{k_0 + \cdots + k_\nu}{k_0, \cdots, k_\nu} \times \\ &\times \binom{(k_0 + \cdots + k_\nu) + k_{\nu+1} + \cdots + k_{n-1}}{k_0 + \cdots + k_\nu, k_{\nu+1}, \cdots, k_{n-1}} , \end{aligned}$$

for any $\nu = 1,\cdots,n-2$. For $\nu = 1$, the first factor on the right is the binomial coefficient

$$\binom{k_0 + k_1}{k_0}$$

Iterating this case of (4.2) in the second factor on the right, and putting

$$\begin{aligned} k'_0 &= k_0 \\ k'_1 &= k_0 + k_1 \\ &\vdots \\ k'_{n-1} &= k_0 + k_1 + \cdots + k_{n-1} , \end{aligned} \tag{4.3}$$

one obtains the reduction

$$\binom{k_0 + \cdots + k_{n-1}}{k_0 , \cdots , k_{n-1}} = \binom{k'_1}{k'_0}\binom{k'_2}{k'_1}\cdots\binom{k'_{n-1}}{k'_{n-2}}. \tag{4.4}$$

Application of this reduction to our final result (3.5) yields the equivalent expression

$$\begin{aligned} p_k^{(\nu)} = \sum_{S(\mathfrak{M})=k+(n-\nu)} &\binom{k'_1}{k'_0}\binom{k'_2}{k'_1}\cdots\binom{k'_{n-1}}{k'_{n-2}} \times \\ &\times \frac{k'_\nu}{k'_{n-1}} a_0^{k'_0} a_1^{k'_1-k'_0} \cdots a_{n-1}^{k'_{n-1}-k'_{n-2}} \quad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix}\right) \end{aligned} \tag{4.5}$$

where

$$S(\mathfrak{M}) = k'_0 + \cdots + k'_{n-1} \tag{4.6}$$

is the simpler linear form obtained by the transformation (4.3) from $L(\mathfrak{M})$ in (2.2). The silent summation condition $k_0,\cdots,k_{n-1} \geq 0$ is transformed in $0 \leq k'_0 \leq \cdots \leq k'_{n-1}$.

Special cases of the formulae (4.5), with $n = 2$ and $n = 3$, have recently been developed by Arkin-Hoggatt [4].

5. We now turn to the general case of an arbitrary period length ℓ. To the ℓ recurrency formula (0.5), viz.,

$$p_{k\ell+\lambda}^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} \qquad \left(\begin{matrix} k \geq 0 \\ \lambda = 0, \cdots, n-1 \end{matrix}\right) \tag{5.1}$$

with the initial conditions (0.4), viz.,

$$p_{-(n-\nu')}^{(\nu)} = e_{\nu'}^{(\nu)} \qquad (\nu, \nu' = 0, \cdots, n-1) \tag{5.2}$$

we let correspond the ℓ polynomials

$$F^{(\lambda)} = F^{(\lambda)}(x) = 1 - \sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} x^{n-\nu'}$$

and the n generating functions

$$p^{(\nu)} = p^{(\nu)}(x) = \sum_{k \geq 0} p^{(\nu)} x^k .$$

We split these polynomials and functions into components, corresponding to the residue classes mod ℓ of the x-exponents:

$$\begin{aligned} F^{(\lambda)} &= \sum_{\lambda'=0}^{\ell-1} F_{\lambda'}^{(\lambda)} \qquad \text{with} \qquad F_{\lambda'}^{(\lambda)} = F_{\lambda'}^{(\lambda)}(x) \\ &= e_{\lambda'}^{(0)} - \sum_{\substack{\nu'=0 \\ n-\nu' \equiv \lambda' \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} x^{n-\nu'} , \end{aligned} \tag{5.3}$$

$$p^{(\nu)} = \sum_{\lambda''=0}^{\ell-1} p_{\lambda''}^{(\nu)} \quad \text{with} \quad p_{\lambda''}^{(\nu)} = p_{\lambda''}^{(\nu)}(x) =$$

(5.4)

$$= \sum_{k \geq 0} p_{k\ell+\lambda''}^{(\nu)} x^{k\ell+\lambda''} .$$

In order to translate the recurrency formulae (5.1) with the initial conditions (5.2) into algebraic expressions for the generating functions, we multiply, for each fixed λ and ν, the terms $a_{\nu'}^{(\lambda)} x^{n-\nu'}$ of a component $F_{\lambda'}^{(\lambda)}$ by that component $p_{\lambda''}^{(\nu)}$ for which

(5.5) $$\lambda' + \lambda'' \equiv \lambda \mod \ell .$$

Subsequently we sum up, first over the ν' with

(5.6) $$n - \nu' \equiv \lambda' \mod \ell ,$$

and then over the ℓ pairs λ', λ'' with (5.5). According to the congruences (5.5) and (5.6), we put

(5.7) $$(n - \nu') + \lambda'' = \lambda + h\ell ,$$

with an integer $h \geq 0$. The whole procedure will be quite analogous to that in Section 1 for the special case $\ell = 1$. In the first place, one has

$$a_{\nu'}^{(\lambda)} x^{n-\nu'} p_{\lambda''}^{(\nu)} = \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{k\ell+\lambda''}^{(\nu)} x^{(k\ell+\lambda'')+(n-\nu')}$$

$$= \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{k\ell+\lambda''}^{(\nu)} x^{(k+h)\ell+\lambda} \qquad \text{(by (5.7))}$$

$$= \sum_{k \geq h} a_{\nu'}^{(\lambda)} p_{(k-h)\ell+\lambda''}^{(\nu)} x^{k\ell+\lambda}$$

$$= \sum_{k \geq h} a_{\nu'}^{(\lambda)} p_{(k-h)\ell+\lambda'}^{(\nu)} x^{k\ell+\lambda} \qquad \text{(by (5.7))}$$

$$= \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$- \sum_{0 \leq k \leq h-1} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$= \sum_{k \geq 0} a_{\nu'}^{(\lambda)} p_{(k\ell+\lambda)-(n-\nu')}^{(\nu)} x^{k\ell+\lambda}$$

$$- \begin{Bmatrix} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} & \text{for } \nu' \equiv \nu - \lambda \bmod \ell \text{ and } \nu' \leq \nu \\ 0 \text{ otherwise} & \end{Bmatrix} .$$

The latter one sees as follows. The summation condition $0 \leq k \leq h - 1$ implies, again by (5.7), the inequality chain

$$-n \leq -(n - \nu') \leq \lambda - (n - \nu') \leq (k\ell + \lambda) - (n - \nu') \leq ((h - 1)\ell + \lambda'')$$
$$- (n - \nu') = -(\ell - \lambda'') \leq -1 ,$$

so that the initial conditions (5.2) are applicable. They say that almost all terms of the sum in question vanish, save only one with

$$(k\ell + \lambda) - (n - \nu') = -(n - \nu), \qquad \text{or else,} \qquad k\ell + \lambda = \nu - \nu' .$$

Such a term can occur only if $\nu' \equiv \nu - \lambda \bmod \ell$ and $\nu' \leq \nu$. If these conditions are satisfied, it actually occurs, because then the equation $k\ell + \lambda = \nu - \nu'$ has a solution $k \geq 0$ with

$$k\ell = (\nu - \nu') - \lambda < (n - \nu') - \lambda = h\ell - \lambda'' \leq h\ell ,$$

and hence $k \leq h - 1$.

Summation over the $\nu' = 0, \cdots, n - 1$ with $n - \nu' \equiv \lambda' \bmod \ell$, according to (5.3) now yields

$$(F_{\lambda'}^{(\lambda)} - e_{\lambda'}^{(0)})p_{\lambda''}^{(\nu)} = -\sum_{k\geq 0}\left(\sum_{\substack{\nu'=0 \\ n-\nu'\equiv\lambda' \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} p_{(\ell+\lambda)-(n-\nu')}^{(\nu)}\right) x^{k\ell+\lambda} +$$

$$+ \sum_{\substack{\nu'=0 \\ n-\nu'\equiv\lambda' \bmod \ell \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{\nu} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} ,$$

and summation over the pairs λ', λ'' with $\lambda' + \lambda'' \equiv \lambda \bmod \ell$ further yields

$$\sum_{\lambda'+\lambda''\equiv \bmod \ell} F_{\lambda'}^{(\lambda)} P_{\lambda''}^{(\nu)} - P_{\lambda}^{(\nu)} = -\sum_{k\geq 0}\left[\sum_{\nu'=0}^{n-1} a_{\nu'}^{(\lambda)} p_{(\ell+\lambda)-(n-\nu')}^{(\nu)}\right] \times$$

$$\times\ x^{k\ell+\lambda} + \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} .$$

Here the negative terms on the left and right are equal to each other on account of the recurrency formulae (5.1). Thus the following system of ℓ linear equations for the ℓ components $P_{\lambda''}^{(\nu)}$ of the generating function $P^{(\nu)}$ results:

$$\sum_{\lambda'+\lambda''\equiv\lambda \bmod \ell} F_{\lambda'}^{(\lambda)} P_{\lambda''}^{(\nu)} = A^{(\lambda,\nu)} \quad \text{with} \quad A^{(\lambda,\nu)}$$

(5.8)

$$= A^{(\lambda,\nu)}(x) = \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv\lambda \bmod \ell}}^{n-1} a_{\nu'}^{(\lambda)} x^{\nu-\nu'} .$$

The matrix of its coefficients is built up from the components $F_{\lambda'}^{(\lambda)}$ of the characteristic polynomials $F^{(\lambda)}$. Lines and columns of this matrix are specified by λ and $\lambda'' \equiv \lambda - \lambda' \bmod \ell$ (not by λ and λ'). Written out fully, it is the matrix

$$\left(F^{(\lambda)}_{\lambda-\lambda''}\right)_{\substack{\lambda \text{ lines} \\ \lambda'' \text{ columns}}} = \begin{pmatrix} F^{(0)}_0 & F^{(0)}_{\ell-1} & \cdots & F^{(0)}_1 \\ F^{(1)}_1 & F^{(1)}_0 & \cdots & F^{(1)}_2 \\ \vdots & & & \\ F^{(\ell-1)}_{\ell-1} & F^{(\ell-1)}_{\ell-2} & \cdots & F^{(\ell-1)}_0 \end{pmatrix} .$$

Here $\lambda - \lambda''$ on the left is to be understood as reduced to its least non-negative residue mod ℓ.

Now let

$$D = \left| F^{(\lambda)}_{\lambda-\lambda''} \right|_{\substack{\lambda \text{ lines} \\ \lambda'' \text{ columns}}}$$

denote the determinant of this matrix and $(D^{(\lambda)}_{\lambda-\lambda''})$ its transposed adjoined matrix. Then the linear system (5.8) has the solution

$$P^{(\nu)}_{\lambda''} = \frac{\sum_{\lambda=0}^{\ell-1} D^{(\lambda)}_{\lambda-\lambda''} A^{(\lambda,\nu)}}{D} , \qquad (\lambda'' = 0, \cdots, \ell - 1) . \tag{5.9}$$

From this one obtains the following algebraic expressions for the generating functions themselves:

$$P^{(\nu)} = \frac{\sum_{\lambda=0}^{\ell-1} \left(\sum_{\lambda''=0}^{\ell-1} D^{(\lambda)}_{\lambda-\lambda''} \right) A^{(\lambda,\nu)}}{D} , \qquad (\nu = 0, \cdots, n - 1) . \tag{5.10}$$

In order to obtain explicit expressions for the recurrent sequences $p^{(\nu)}$, one has to develop these rational functions of x into power series in x. This seems however extremely difficult. One would first have to find a sufficiently smooth expression for the determinant D and its minors $D^{(\lambda)}_{\lambda-\lambda''}$.

In the following two sections we illustrate this on the next-simplest case $\ell = 2$ and carry it through to the end under the special assumption $n = 2$. After what has been delineated in the preceding sections, we can be brief in doing this.

6. In the special case $\ell = 2$ we have to consider two alternating recurrency formulae

$$p_{2k}^{(\nu)} = \sum_{\nu'=0}^{n-1} a_{\nu'} p_{2k-(n-\nu')}^{(\nu)} ,$$

$$p_{2k+1}^{(\nu)} = \sum_{\nu'=0}^{n-1} b_{\nu'} p_{(2k+1)-(n-\nu')}^{(\nu)} ,$$

for each of the n linearly independent initial conditions

$$p_{-(n-\nu')}^{(\nu)} = e_{\nu'}^{(\nu)} \qquad (\nu, \nu' = 0, \cdots, n-1) .$$

For the sake of easier readability, we here have distinguished the two coefficient sequences, hitherto denoted by $a_{\nu'}^{(\lambda)}$, instead by the upper indices $\lambda = 0, 1$ by writing them with two different letters a, b. In the same manner we denote the polynomial pairs $F^{(\lambda)}$ and $A^{(\lambda,\nu)}$ $(\lambda = 0, 1)$ now by F, G and $A^{(\nu)}$, $B^{(\nu)}$, respectively.

From the pair of characteristic polynomials

$$F = F(x) = 1 - \sum_{\nu'=0}^{n-1} a_{\nu'} x^{n-\nu'} = F_0 + F_1 ,$$

$$G = G(x) = 1 - \sum_{\nu'=0}^{n-1} b_{\nu'} x^{n-\nu'} = G_0 + G_1 ,$$

each decomposed in its even and odd components, algebraic expressions for the generating functions

$$P^{(\nu)} = P^{(\nu)}(x) = \sum_{k \geq 0} p_{\kappa}^{(\nu)} x^k = P_0^{(\nu)} + P_1^{(\nu)} ,$$

likewise decomposed, are found as follows.

The linear equation pair (5.8) for the component pair $P_0^{(\nu)}$, $P_1^{(\nu)}$ has the matrix

$$\begin{pmatrix} F_0 & F_1 \\ G_1 & G_0 \end{pmatrix}$$

with the determinant

$$D = F_0G_0 - F_1G_1 \quad ,$$

and with the transposed adjoined matrix

$$\begin{pmatrix} G_0 & -F_1 \\ -G_1 & F_0 \end{pmatrix} \quad .$$

The terms on the right are

$$A^{(\nu)} = \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv 0 \bmod 2}}^{\nu} a_{\nu'} x^{\nu-\nu'}$$

$$B^{(\nu)} = \sum_{\substack{\nu'=0 \\ \nu-\nu'\equiv 1 \bmod 2}}^{\nu} b_{\nu'} x^{\nu-\nu'} \quad .$$

Hence the solution (5.9) for the components is

$$P_0^{(\nu)} = \frac{G_0A^{(\nu)} - F_1B^{(\nu)}}{F_0G_0 - F_1G_1}, \qquad P_1^{(\nu)} = \frac{-G_1A^{(\nu)} + F_0B^{(\nu)}}{F_0G_0 - F_1G_1} \quad ,$$

and the generating functions (5.10) themselves are

$$P^{(\nu)} = \frac{(G_0 - G_1)A^{(\nu)} + (F_0 - F_1)B^{(\nu)}}{F_0G_0 - F_1G_1} \quad . \tag{6.1}$$

It is worth remarking that this can be written in such a way that only the characteristic polynomials F, G themselves, not their components, figure in it. For, the component pairs are given by

$$F_0(x) = \frac{F(x) + F(-x)}{2}, \qquad F_1(x) = \frac{F(x) - F(-x)}{2},$$

$$G_0(x) = \frac{G(x) + G(-x)}{2}, \qquad G_1(x) = \frac{G(x) - G(-x)}{2},$$

Thus the determinant becomes

$$D(x) = \frac{F(x)G(-x) + F(-x)G(x)}{2},$$

and the generating functions become

$$P^{(\nu)}(x) = \frac{G(-x)A^{(\nu)}(x) + F(-x)B^{(\nu)}(x)}{D(x)} . \tag{6.2}$$

7. Under the special assumption $n = 2$, one has

$$\begin{aligned}
F &= 1 - a_1x - z_0x^2 = (1 - a_0x^2) - a_1x, \\
G &= 1 - b_1x - b_0x^2 = (1 - b_0x^2) - b_1x, \\
\end{aligned}$$

$$\begin{array}{rl|rl}
A^{(0)} & = a_0 & A^{(1)} & = a_1 \\
B^{(0)} & = 0 & B^{(1)} & = b_0x,
\end{array}$$

$$\begin{aligned}
D &= (1 - a_0x^2)(1 - b_0x^2) - a_1b_1x^2 \\
&= 1 - (a_0 + b_0 + a_1b_1)x^2 + a_1b_1x^4,
\end{aligned}$$

$$P^{(0)} = \frac{a_0 + a_0b_1x - a_0b_0x^2}{D}, \qquad P^{(1)} = \frac{a_1 + (b_0 + a_1b_1)x - a_0b_0x^3}{D}.$$

The power series development of $1/D$ is

$$\begin{aligned}\frac{1}{D} &= \sum_{k\geq 0} \left((a_0 + b_0 + a_1b_1)x^2 - a_0b_0x^4 \right)^k \\ &= \sum_{k_0, k_1 \geq 0} \binom{k_0 + k_1}{k_1} (a_0 + b_0 + a_1b_1)^{k_0}(-a_0b_0)^{k_1} x^{2k_0+4k_1} \\ &= \sum_{k\geq 0} \sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0 + k_1}{k_1}(a_0 + b_0 + a_1b_1)^{k_0}(a_0b_0)^{k_1} \quad x^{2k} .\end{aligned}$$

From this, one obtains easily the following power series developments for the even and odd components of the two generating functions:

$$(7.1)\quad \begin{cases} P_0^{(0)} = \dfrac{a_0 - a_0b_0x^2}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1+1}b_0^{k_1}\right] x^{2k} \\ \qquad\qquad - \displaystyle\sum_{k\geq 0}\left[\sum_{\substack{k_0+2k_1=k\\ (k_0\geq 1)}} (-1)^{k_1}\binom{k_0+k_1-1}{k_1}(a_0+b_0+a_1b_1)^{k_0-1}a_0^{k_1+1}b_0^{k_1+1}\right] x^{2k} \\ P_1^{(0)} = \dfrac{a_0b_1}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}b_1(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1+1}b_0^{k_1}\right] x^{2k+1} \\ P_0^{(1)} = \dfrac{a_1}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1}\binom{k_0+k_1}{k_1}a_1(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1}b_0^{k_1}\right] x^{2k} \\ P_1^{(1)} = \dfrac{(b_0 + a_1b_1)x - a_0b_0x^3}{D} = \displaystyle\sum_{k\geq 0}\left[\sum_{k_0+2k_1=k} (-1)^{k_1} \times\right. \\ \qquad\qquad \left.\times\binom{k_0+k_1}{k_1}(b_0+a_1b_1)(a_0+b_0+a_1b_1)^{k_0}a_0^{k_1}b_0^{k_1}\right] x^{2k+1} \\ \qquad\qquad - \displaystyle\sum_{k\geq 0}\left[\sum_{\substack{k_0+2k_1=k\\ (k_1\geq 1)}} (-1)^{k_1}\binom{k_0+k_1-1}{k_1}\times\right. \\ \qquad\qquad \left.\times (a_0+b_0+a_1b_1)^{k_0-1}a_0^{k_1+1}b_0^{k_1+1}\right] x^{2k+1} . \end{cases}$$

The sums in square brackets — or in the first and fourth cases, more exactly, their differences — are the looked for explicit expressions for the recurrent sequences

$$p_{2k}^{(0)}, \quad p_{2k+1}^{(0)} \quad \text{and} \quad p_{2k}^{(1)}, \quad p_{2k+1}^{(1)} .$$

8. We finally come to consider the important special case, where $\ell = n$, i.e., the period length coincides with the dimension of the algorithm, and where n^2 indeterminate recurrency coefficients $a_{\nu'}^{(\lambda)}$ are specialized to combinations of only $n+1$ indeterminates $a_{\nu'}$ and t as specified in (0.10), (0.11), viz.,

$$(8.1)\qquad a_{\nu'}^{(\lambda)} = t^{d_n(\lambda,\nu')} a_{\nu'} \quad \text{with} \quad d_n(\lambda,\nu') = \begin{cases} 0 & \text{for } \lambda + \nu' < n \\ 1 & \text{for } \lambda + \nu' \geq n \end{cases}.$$

In this case the recurrency formulae (0.5) specialize to

$$(8.2)\qquad p_{kn+\lambda}^{(\nu)} = \sum_{\nu'=0}^{n-1} t^{d_n(\lambda,\nu')} a_{\nu'} p_{(kn+\lambda)-(n-\nu')}^{(\nu)} \quad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix} \right),$$

with the n linearly independent initial conditions (0.4), viz.,

$$(8.3)\qquad p_{-(n-\nu')}^{(\nu)} = e_{\nu'}^{(\nu)} \qquad (\nu,\nu' = 0,\cdots,n-1).$$

These recurrency formulae can be reduced to those of the special case $t = 1$, but with new coefficients. For this purpose consider the modified sequences

$$(8.4)\qquad \bar{p}_{kn+\lambda}^{(\nu)} = t^{k+1} p_{kn+\lambda}^{(\nu)}.$$

They satisfy again the initial conditions (8.3). Now the $p^{(\nu)}$-subscripts on the right of (8.2) reduce as follows to the canonical form on the left:

$$(kn + \lambda) - (n - \nu') = (k - 1)n + (\lambda + \nu') = (k - 1 + d_n(\lambda,\nu'))n + \lambda'$$

with $0 \leq \lambda' \leq n - 1$.

Hence,

$$p^{-(\nu)}_{(kn+\lambda)-(n-\nu')} = t^{k+d_n(\lambda,\nu')} p^{(\nu)}_{(kn+\lambda)-(n-\nu')} \quad . \tag{8.5}$$

From (8.4), (8.5), we obtain the following transformation of the recurrency formulae (8.2):

$$\begin{aligned} p^{-(\nu)}_{kn+\lambda} &= t^{k+1} p^{(\nu)}_{kn+\lambda} \\ &= \sum_{\nu'=0}^{n-1} t^{k+1+d_n(\lambda,\nu')} a_{\nu'} p^{(\nu)}_{(\;n+\lambda)-(n-\nu')} \\ &= \sum_{\nu'=0}^{n-1} t a_{\nu'} p^{-(\nu)}_{(\;n+\lambda)-(n-\nu')} \quad . \end{aligned}$$

Thus the modified sequences $p^{-(\nu)}_{kn+\lambda}$ satisfy the linear recurrency (0.6) with the modified coefficients $ta_{\nu'}$, and, as already said, with the same initial conditions (0.4). According to (0.7), they are therefore given explicitly by

$$\begin{aligned} p^{-(\nu)}_{kn+\lambda} &= \sum_{L(\mathfrak{M})=(kn+\lambda)+(n-\nu)} \binom{k_0 + \cdots + k_n}{k_0\,,\ \cdots\ ,\ k_n} \times \\ &\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} t^{k_0+\cdots+k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \\ &\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix} \right) , \end{aligned} \tag{8.6}$$

Going back to the original sequences $p^{(\nu)}_{nk+\lambda}$ by (8.4) and replacing the no longer necessary detailed subscripts $nk+\lambda$ by simply k, we obtain our second chief result,

$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{L(\mathfrak{M})=k+(n-\nu)} \binom{k_0 + \cdots + k_n}{k_0, \cdots, k_n} \times$$

$$\times \frac{k_0 + \cdots + k_\nu}{k_0 + \cdots + k_{n-1}} t^{k_0 + \cdots + k_{n-1}} a_0^{k_0} \cdots a_{n-1}^{k_{n-1}} \tag{8.7}$$

$$\begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix}$$

as announced in (0.12). The remark after (3.5), concerning validity even for $k \geq -n$, i.e., including the initial values holds obviously for (8.7) as well.

Application of the reduction (4.3), (4.4) of polynomial to binomial coefficients to this result yields, in analogy to (4.5), the equivalent expression

$$p_k^{(\nu)} = t^{-\left[\frac{k}{n}\right]-1} \sum_{S(\mathfrak{M}) = +(n-\nu)} \binom{k'_1}{k'_0}\binom{k'_2}{k'_1} \cdots \binom{k'_{n-1}}{k'_{n-2}} \times$$

$$\times \frac{k'_\nu}{k'_{n-1}} t^{k'_{n-1}} a_0^{k'_0} a_1^{k'_1 - k'_0} \cdots a_{n-1}^{k'_{n-1} - k'_{n-2}} \tag{8.8}$$

$$\begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{pmatrix}$$

CHAPTER II. GENERALIZED FIBONACCI NUMBERS WITH TIME IMPULSES

9. It is known from the history of mathematics [5] that the original Fibonacci numbers F_k, named after their discoverer, and defined by the recurrency formula

$$F_{k+2} = F_k + F_{k+1} \qquad (k \geq 1) \tag{9.1}$$

with the initial values

$$F_1 = 1, \qquad F_2 = 1, \tag{9.2}$$

describe the mathematical structure of a biological process in nature, viz., of the way rabbits would multiply if no outside factors would interfere with this idealized fertility. From a purely speculative viewpoint this recurrency definition could be replaced by a variety of other structures. So, for instance, the initial values could be replaced by others, as was done by E. Lucas. Thus (9.2) by (in new notation) becomes

$$L_1 = 1, \qquad L_2 = 3. \tag{9.2'}$$

Or the dimension 2 of the recurrency could be increased to any $n \geq 2$, as was done by the first author [3] who substituted (9.1), (9.2) by

$$\overset{n}{F}_{k+n} = \overset{n}{F}_k + \cdots + \overset{n}{F}_{k+(n-1)} \qquad (k \geq 1), \tag{9.3}$$

$$\overset{n}{F}_1, \cdots, \overset{n}{F}_{n-1} = 0, \qquad \overset{n}{F}_n = 1 . \tag{9.4}$$

This generalization to higher dimension could be carried further by considering recurrencies with constant weights $a_0, \cdots, a_{n-1}$ given to the preceding terms, viz.,

$$\overset{n}{F}_{k+n} = a_0 \overset{n}{F}_k + \cdots + a_{n-1} \overset{n}{F}_{k+n-1} \qquad (k \geq 1) \tag{9.5}$$

with arbitrary initial values

$$\overset{n}{F}_1, \cdots, \overset{n}{F}_n .$$

Formula (9.5) is actually the recurrency law (0.6) of our introductory section.

The question which is the natural generalization of the original Fibonacci numbers is idle. The answer to it depends on the viewpoint one takes and is a matter of mathematical taste and preferences. Raney [6], for instance, has proposed a generalization widely different in viewpoint and preferences from those mentioned above.

From a purely biological, or even mechanical, viewpoint one would rather expect that a process in nature, depending on n preceding positions, would <u>not</u> go on with such an idealized uniform law of passing to the next position as are those mentioned above, but rather with additional impulses, acting on this law, which are themselves functions of time. It is already a daring presumption that such impulses, imposed by nature, would be recurring regularly. But the purely mathematical applications which will be given in a subsequent paper are some justification for the subsequent new, and in the view of the authors, more "natural" generalization.

For this proposed <u>generalization of the Fibonacci numbers</u> we modify the recurrency law (9.5), i.e., (0.6) by <u>time impulses</u> in the shape of a constant time factor $t \neq 0$, attached to some of the weights $a_0, \cdots, a_{n-1}$ according to the more general recurrency law (0.5) of our introductory section. As initial values we admit throughout the n linearly independent standard sets (0.4).

From them any set of n initial values may be linearly combined, and the corresponding recurrent sequence will then be obtained from those corresponding to (0.4) by the same linear combination.

10. Before we apply the general results (8.7), (8.8) of our first chapter to special cases of the generalized Fibonacci numbers with time impulses, let us make some preliminary remarks.

1.) The restriction of summation

$$L(\mathfrak{M}) = nk_0 + (n-1)k_1 + \cdots + 1k_{n-1} = k + (n - \nu)$$

in the sums (8.7) with multinomial coefficients

$$\binom{k_0 + \cdots + k_{n-1}}{k_0, \cdots, k_{n-1}}$$

can be removed by eliminating the last summation variable k_{n-1} (the only one with coefficient 1) on the strength of that restriction, viz., by putting

$$k_{n-1} = k + (n - \nu) - (nk_0 + \cdots + 2k_{n-2}) \tag{10.1}$$

wherever k_{n-1} occurs in the terms of the sum. It is convenient to combine this elimination with the reduction (4.2) of the multinomial coefficients of order n to such of order n - 1 and binomial coefficients. Thus the formulae (8.7) become

$$\begin{aligned} p_\kappa^{(\nu)} = \sum_{k_0,\cdots,k_{n-2}} &\binom{S_0(\mathfrak{M})}{k_0,\cdots,k_{n-2}}\binom{k + (n-\nu) - L_0(\mathfrak{M})}{S_0(\mathfrak{M})} \times \\ &\times \frac{k_0 + \cdots + k_\nu}{k + (n - \nu) - L_0(\mathfrak{M})} \times \\ &\times t^{k - \left[\frac{k}{n}\right] + (n-\nu) - 1 - L_0(\mathfrak{M})} \times \\ &\times a_0^{k_0} a_1^{k_1} \cdots a_{n-2}^{k_{n-2}} a_{n-1}^{k+(n-\nu)-L_0(\mathfrak{M})-S_0(\mathfrak{M})} . \end{aligned} \tag{10.2}$$

$$\begin{pmatrix} k \geq 0 \\ \nu = 0, \cdots, n-2 \end{pmatrix}$$

$$(10.2')\left\{\begin{aligned} p^{(n-1)} &= \sum_{k_0,\cdots,k_{n-2}} \binom{S_0(\mathfrak{M})}{k_0,\cdots,k_{n-2}}\binom{k+1-L_0(\mathfrak{M})}{S_0(\mathfrak{M})}\times \\ &\times t^{k-\left[\frac{k}{n}\right]-L_0(\mathfrak{M})}\times \\ &\times a_0^{k_0}a_1^{k_1}\cdots a_{n-2}^{k_{n-2}}a_{n-1}^{k+1-L_0(\mathfrak{M})-S_0(\mathfrak{M})} \\ &\qquad\qquad \binom{k \geq 0}{\nu = n-1}, \end{aligned}\right.$$

with the reduced linear forms

$$L_0(\mathfrak{M}) = (n-1)k_0 + \cdots + 1k_{n-2},\quad S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2}.$$

For confirmation of (10.2), (10.2'), notice that with the help of these two linear forms the substitution (10.1) takes the form

$$k_{n-1} = k + (n-\nu) - L_0(\mathfrak{M}) - S_0(\mathfrak{M}).$$

Notice further that the silent summation condition $k_{n-1} \geq 0$ is transformed into the upper limitation of summation

$$L_0(\mathfrak{M}) + S_0(\mathfrak{M}) \leq k + (n-\nu).$$

This limitation may be passed over silently by the following conventions. For $L_0(\mathfrak{M}) < k + (n-\nu)$ no convention is necessary, because in this case the binomial coefficient vanishes if $S_0(\mathfrak{M}) > k + (n-\nu) - L_0(\mathfrak{M})$; in particular for $L_0(\mathfrak{M}) = k + (n-\nu)$, however, we convene to consider the denominator of the subsequent fraction cancelled against the same factor of the factorial in the numerator of the binomial coefficient, as will actually be done later. For $L_0(\mathfrak{M}) > k + (n-\nu)$, we convene to consider the binomial coefficient as being 0; this is <u>not</u> in accordance with the <u>usual</u> extension of Pascal's triangle to negative "numerators"-k by means of the fundamental recurrency property, fixing arbitrarily,

$$\binom{-k}{0} = 1 ,$$

since this extension gives them non-zero values as long as the "denominator" is non-negative.

Observe, by the way, that for $\nu = n - 2$ one has $k_0 + \cdots + k_\nu = S_0(\mathfrak{M})$. Hence in this case the binomial coefficient can be combined with the subsequent fraction to

$$\binom{k + (n - \nu) - L_0(\mathfrak{M}) - 1}{S_0(\mathfrak{M}) - 1} .$$

In (10.2), (10.2'), the restriction of summation $L(\mathfrak{M}) = k + (n - \nu)$ has disappeared. This is deceptive, however, in cases where the recurrency coefficient a_{n-1} is specialized to 0. For, in such cases only the terms in which a_{n-1} has exponent $k_{n-1} = 0$ remain in the sum. Thus the restriction reappears, so to say, by the backdoor, in a slightly modified form, viz., without the term $1k_{n-1}$. This is a change to the worse, even to the worst, into the bargain since now there is no longer a term with coefficient 1 which would allow a further elimination.

2.) Things stand better with the sums (8.8), in which the polynomial coefficients have been reduced to products

$$\binom{k_1'}{k_0'} \cdots \binom{k_{n-1}'}{k_{n-2}'}$$

of binomial coefficients. Here, in the restriction of summation

$$S(\mathfrak{M}) = k_0' + \cdots + k_{n-1}' = k + (n - \nu) ,$$

<u>each</u> of the n summation variables $k_0', \cdots, k_{n-1}'$ has coefficient 1, so that there are n different ways of removing the restriction by elimination. However, in cases where a recurrency coefficient $a_{\nu'}$ with $\nu' \geq 1$ is specialized to 0, only the terms with $k_{\nu'}' = k_{\nu'-1}'$ remain in the sum, so that the coefficient of $k_{\nu'}'$ becomes higher than 1, and thus elimination of $k_{\nu'}'$ is barred. For this reason the restriction can be removed only in cases where either at least one consecutive pair $a_{\nu'}, a_{\nu'+1}$ with $0 \leq \nu' \leq n - 2$ or a_{n-1} alone is not specialized to 0.

We shall chiefly be concerned with the latter case $a_{n-1} \neq 0$, in which reduction of the sums (8.7) to unrestricted summation has already been achieved in (10.2), (10.2'). For treating the cases where some of the preceding $a_{\nu'}$ are specialized to 0, it will, however, be more convenient to start from the corresponding reduction of the sums (8.8), viz.,

$$(10.3)\quad \left\{\begin{aligned} p_\kappa^{(\nu)} = \sum_{k'_0,\cdots,k'_{n-2}} &\binom{k'_1}{k'_0}\cdots\binom{k'_{n-2}}{k'_{n-3}}\binom{k+(n-\nu)-S_0(\mathfrak{M}')}{k'_{n-2}}\times\\ &\times\frac{k'_\nu}{k+(n-\nu)-S_0(\mathfrak{M}')}\times\\ &\times t^{k-\left[\frac{k}{n}\right]+(n-\nu)-1-S_0(\mathfrak{M}')}\times\\ &\times a_0^{k'_0}a_1^{k'_1-k'_0}\cdots a_{n-2}^{k'_{n-2}-k'_{n-3}}\times\\ &\times a_{n-1}^{k+(n-\nu)-S_0(\mathfrak{M}')-k'_{n-2}}\\ &\qquad\qquad\left(\begin{aligned}k &\geq 0\\ \nu &= 0,\cdots,n-2\end{aligned}\right)\end{aligned}\right.$$

$$(10.3')\quad \left\{\begin{aligned} p_\kappa^{(n-1)} = \sum_{k'_0\cdots k'_{n-2}} &\binom{k'_1}{k'_0}\cdots\binom{k'_{n-2}}{k'_{n-1}}\binom{k+1-S_0(\mathfrak{M}')}{k'_{n-2}}\times\\ &\times t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')}\times\\ &\times a_0^{k'_0}a_1^{k'_1-k'_0}\cdots a_{n-2}^{k'_{n-2}-k'_{n-3}}\times\\ &\times a_{n-1}^{k+1-S_0(\mathfrak{M}')-k'_{n-2}},\\ &\qquad\qquad\left(\begin{aligned}k &\geq 0\\ \nu &= n-1\end{aligned}\right)\end{aligned}\right.$$

with the reduced linear form

$$S_0(\mathfrak{M}') = k'_0 + \cdots + k'_{n-2}$$

The remark made after (10.2), (10.2') about the silent summation condition $k'_{n-1} \geq 0$ holds, mutatis mutandis, also for the silent summation condition $k'_{n-1} \geq k'_{n-2}$ in (10.3), (10.3'), the latter corresponding to the former under the transformation (4.3). We uphold the conventions made in that remark.

We must enlarge, however, on the subsequent observation about the possibility of combining the binomial coefficient in (10.2) with the subsequent fraction for $\nu = n - 2$, because this observation generalizes here to all $\nu = 0, \cdots, n - 2$ and thus allows to get rid of these fractions altogether. This is seen by the following chain of reductions:

$$\binom{k + (n - \nu) - S_0(\mathfrak{M}')}{k'_{n-2}} \frac{k'_\nu}{k + (n - \nu) - S_0(\mathfrak{M}')} = \frac{k'_\nu}{k'_{n-2}} \binom{k + (n - \nu) - S_0(\mathfrak{M}') = 1}{k'_{n-2} - 1} \ .$$

$$\binom{k'_{n-2}}{k'_{n-3}} \frac{k'_\nu}{k'_{n-2}} = \frac{k'_\nu}{k'_{n-3}} \binom{k'_{n-2} - 1}{k'_{n-3} - 1}$$

$$\vdots$$

$$\binom{k'_{\nu+1}}{k'_\nu} \frac{k'_{\nu+1}}{k'_\nu} = \binom{k'_{\nu+1} - 1}{k'_\nu - 1} \ .$$

which, of course, has to be considered only for $k'_\nu \geq 1$ and hence all subsequent $k'_{\nu+1}, \cdots, k'_{n-2} \geq 1$, too. This chain of reduction yields

$$\binom{k'_{\nu+1}}{k'_\nu} \cdots \binom{k'_{n-2}}{k'_{n-3}} \binom{k + (n - \nu) - S_0(\mathfrak{M}')}{k'_{n-2} - 1} \frac{k'_\nu}{k + (n - \nu) - S_0(\mathfrak{M}')}$$
$$= \binom{k'_{\nu+1} - 1}{k'_\nu - 1} \cdots \binom{k'_{n-2} - 1}{k'_{n-3} - 1} \binom{k + (n - \nu) - 1 - S_0(\mathfrak{M}')}{k'_{n-2} - 1}$$

By the transformation

$$k'_\nu - 1 \rightarrow k'_\nu, \cdots, k'_{n+2} - 1 \rightarrow k'_{n-2} \ ,$$

after which the summation range is again $k'_{\nu+1}, \cdots, k'_{n-2} \geq 0$, then

$$S(\mathfrak{M}') \qquad S(\mathfrak{M}') + (n - \nu) - 1 ,$$

and thus (10.3) becomes

$$
(10.4)\quad \left\{
\begin{aligned}
p_k^{(\nu)} = \sum_{k'_0,\cdots,k'_{n-2}} & \binom{k'_1}{k'_0}\cdots\binom{k'_\nu + 1}{k'_{\nu-1}}\cdots\binom{k'_{n-2}}{k'_{n-3}}\times \\
& \times\binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M}')}{k'_{n-2}} t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')}\times \\
& \times a_0^{k'_0+e_0^{(\nu)}} a_1^{k'_1-k'_0}\cdots a_\nu^{k'_{\nu+1}-k'_{\nu-1}}\cdots\times \\
& \times a_{n-2}^{k'_{n-2}-k'_{n-3}} a_{n-1}^{k+e_{n-1}^{(\nu)}-S_0(\mathfrak{M}')-k'_{n-2}} \\
& \qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix}\right)
\end{aligned}
\right.
$$

where the modified middle terms

$$\binom{k'_\nu + 1}{k'_{\nu-1}}$$

and

$$a_\nu^{k'_\nu+1-k'_{\nu-1}}$$

are only meant for $\nu = 1,\cdots,n-2$, and where $e_0^{(\nu)}, e_{n-1}^{(\nu)}$ are the coefficients in the first and last column of the unit matrix, introduced in (0.4); by inserting e_{n-1} at the two places, the case $\nu = n-1$, split off in (10.2'), (10.3'), could now be re-included. Formulae (10.4) could be expressed more concisely introducing also the other $e_{\nu'}^{(\nu)}$ $(\nu' = 1,\cdots,n-1)$ and using the product sign:

$$(10.5)\quad \left\{\begin{aligned} p_k^{(\nu)} &= \sum_{k'_0,\cdots,k'_{n-2}} \left(\prod_{\nu'=0}^{n-2} \binom{k'_{\nu'} + e_{\nu'}^{(\nu)}}{k'_{\nu'-1}} a_{\nu'}^{k'_{\nu'}+e_{\nu'}^{(\nu)}-k'_{\nu'-1}} \right) \times \\ &\times t^{k-\left[\frac{k}{n}\right]-S_0(\mathfrak{M}')} \binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M}')}{k_{n-2}} \times \\ &\times a_{n-1}^{k+e_{n-1}^{(\nu)}-S_0(\mathfrak{M}')-k'_{n-2}} \\ & \qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0,\cdots,n-1 \end{matrix} \right) \end{aligned}\right.$$

where one has to understand formally $k'_{-1} = 0$. For our intention of passing to special cases, though, formulae (10.4) allow a better survey.

Notice that for each $\nu = 0,\cdots,n-2$ the silent summation condition for k'_ν in the formula (10.4) or (10.5) for $p_k^{(\nu)}$ has to be modified into $k'_\nu + 1 \geq k'_{\nu-1}$.

Since the original formulae (3.5), (8.7), (10.2) and (10.2') with the polynomial coefficients will not be referred to again, we shall hence forward simplify the notation by omitting the dashes on $k_1,\cdots,k_{n-2}$.

3.) As to specialization of the recurrency coefficients $a_0, a_1, \cdots, a_{n-1}$, we may suppose without loss of generality $a_0 \neq 0$, by considering only recurrencies of the exact order n. In the Jacobi-Perron algorithm there is always even $a_0 = 1$; see (0.1) and what was explained before and afterwards.

4.) For $a_0 = 1$ and $t = 1$ the two recurrent sequences $p_k^{(0)}$ and $p_k^{(n-1)}$ with the first and last set of our standard initial values (0.4) are <u>essentially equal</u> to each other, i.e., they differ only by a translation of the sequence variable k:

$$(10.6)\qquad p_k^{(n-1)} = p_{k+1}^{(0)} \qquad (k \geq -n) \quad .$$

For, $p_k^{(0)}$ has the initial values, $1,0,\cdots,0$. Hence by the recurrency formula $p_0^{(0)} = a_0 = 1$. Therefore $p_{k+1}^{(0)}$ has the initial values $0,\cdots,0,1$. Since for $t = 1$ the recurrency formulae for $p_k^{(0)}$ and $p_k^{(n-1)}$ are the same, (10.6) follows.

11. We now apply our general results to special cases of the generalized Fibonacci numbers. We base these applications as far as possible on our appropriately adapted main result (10.4) for cases with recurrency coefficient $a_{n-1} \neq 0$. Only in the cases with $a_{n-1} = 0$, treated at the end, we have to go back to the original result (8.8).

1.) The uniform case: $a_0, a_1, \cdots, a_{n-1} = 1$; $t = 1$.

In this case we found it convenient, in order to avoid confusion, to put the recurrency order n on top of the sequence letter, as already done in (9.3-5). Here (10.4) becomes simply

$$\overset{n}{p}{}_k^{(\nu)} = \sum_{k_0,\cdots,k_{n-2}} \binom{k_1}{k_0} \cdots \binom{k_\nu + 1}{k_{\nu-1}} \cdots \binom{k_{n-2}}{k_{n-3}} \times \tag{11.1}$$
$$\times \binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M})}{k_{n-2}} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix}\right),$$

with

$$S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2} \quad .$$

The first and last of these sequences, essentially equal to each other according to (10.6), are essentially equal to the sequence of generalized Fibonacci numbers considered by the first author in his previous paper [3], and mentioned above in (9.3). For, adaptation to the initial values (9.4) of those latter yields

$$\overset{n}{F}_k = \overset{n}{p}{}_{k-n}^{(0)} = \overset{n}{p}{}_{k-(n+1)}^{(n-1)} \qquad (k \geq 1) . \tag{11.2}$$

In particular, for $n = 2$ there remains only one summation variable $k_0 = s$, and (11.1) becomes

$$\overset{2}{p}{}_k^{(\nu)} = \sum_s \binom{k + \nu - s}{s} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, 1 \end{matrix}\right) , \tag{11.3}$$

These two sequences are essentially equal to the sequence (9.1) of the original Fibonacci numbers. For, adaptation to the initial values (9.2) of those latter yields

$$(11.4) \qquad F_k = \overset{2}{p}{}^{(0)}_{k-1} = \overset{2}{p}{}^{(1)}_{k-2}\binom{k-1-s}{s} \qquad (k \geq 1) .$$

Notice that, unfortunately, the initial values (9.4) of the generalized Fibonacci numbers

$$\overset{n}{F}_k$$

are <u>not</u> in accordance with the traditional initial values (9.2) of the original Fibonacci numbers F_k, corresponding to the special case $n = 2$. By (11.2), (11.4) the connection is

$$(11.5) \qquad \overset{2}{F}_{k+1} = F_k ,$$

i.e., a translation by 1. The traditional initial values (9.2) are in accordance with the representation

$$F_k = \frac{\epsilon^k - \epsilon'^k}{\epsilon - \epsilon'} , \qquad (k \geq 0)$$

where

$$\epsilon = \frac{1 + \sqrt{5}}{2} ,$$

whose analogue for the Lucas numbers is

$$L_k = \epsilon^k + \epsilon'^k \qquad (k \geq 0) .$$

The Lucas numbers, according to their initial values (9.2'), are obtained by the linear combination

$$(11.6)\qquad L_k = \overset{2}{p}{}^{(0)}_{k-3} + 3\overset{2}{p}{}^{(1)}_{k-3} = \sum_s \binom{k-3-s}{s} + 3\binom{k-2-s}{s} \qquad (k \geq 3).$$

The representations (11.4) and (11.6) of the historical Fibonacci and Lucas numbers are well known [5].

In all following cases we presuppose

$$a_0 = 1, \qquad t \text{ arbitrary },$$

the latter with the only natural restriction $t \neq 0$.

2.) The multiple uniform case: all $a_1, \cdots, a_{n-1} = a \neq 0$.

In this case we have to attach to the expression (11.1) the powers of t and a according to (10.4). In order to determine the exponent of a in the simplest possible manner, observe that the sum of the exponents of $a_0, a_1, \cdots, a_{n-1}$ in (10.4) (or (10.5)) reduces to $k + 1 - S_0(\mathfrak{M})$. But since here only $a_1, \cdots, a_{n-1} = a$ whereas $a_0 = 1$, the exponent $k_0 + e_0^{(\nu)}$ has to be subtracted. Thus

$$(11.7)\quad \left\{ \begin{aligned} p_k^{(\nu)} &= \sum_{k_0,\cdots,k_{n-1}} \binom{k_1}{k_0} \cdots \binom{k_\nu + 1}{k_{\nu-1}} \cdots \binom{k_{n-2}}{k_{n-3}} \binom{k + e_{n-1}^{(\nu)} - S_0(\mathfrak{M})}{k_{n-2}} \times \\ &\quad \times t^{k - \left[\frac{k}{n}\right] - S_0(\mathfrak{M})} a^{k+1-e_0^{(\nu)} - S_0(\mathfrak{M}) - k_0} \\ &\qquad\qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, \cdots, n-1 \end{matrix} \right) \end{aligned} \right.$$

with

$$S_0(\mathfrak{M}) = k_0 + \cdots + k_{n-2} .$$

We illustrate this by the two lowest cases:

$n = 2$

$$(11.8)\qquad p_k^{(\nu)} = \sum_{k'} \binom{k + \nu - k'}{k'} t^{k - \left[\frac{k}{2}\right] - k'} a^{k-2k'+\nu} \qquad \left(\begin{matrix} k \geq 0 \\ \nu = 0, 1 \end{matrix} \right) ;$$

n = 3

$$
(11.9)\quad \left\{
\begin{aligned}
p_k^{(0)} &= \sum_{k_0,k_1} \binom{k_1}{k_0}\binom{k-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)}\, a^{k-(2k_0+k_1)} \\
p_k^{(1)} &= \sum_{k_0,k_1} \binom{k_1+1}{k_0}\binom{k-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)}\, a^{k+1-(2k_0+k_1)} \\
p_k^{(2)} &= \sum_{k_0,k_1} \binom{k_1}{k_0}\binom{k+1-(k_0+k_1)}{k_1} t^{k-\left[\frac{k}{3}\right]-(k_0+k_1)}\, a^{k+1-(2k_0+k_1)}
\end{aligned}
\right.
$$

$$(k \geq 0) .$$

It would be worthwhile to confirm (11.8) from (7.1) by specializing there $a_0 = 1$, $b_0 = 1$, $a_1 = a$, $b_1 = ta$.

3.) Reduced multiple uniform cases: some $a_{\nu'} = 0$, the other $a_{\nu'} = a \neq 0$ $(\nu' = 1, \cdots, n-1)$.

a) Cases with $a_{n-1} = a \neq 0$.

As we saw in Section 10, in these cases, the general reduction (10.4) to unrestricted summation is effective. The results are obtained from (10.4) by simply adding the summation conditions

$$
\begin{aligned}
k_{\nu'} &= k_{\nu'-1} \quad \text{for all } \nu' \neq \nu \text{ with } a_{\nu'} = 0 , \\
k_\nu &= k_{\nu-1} - 1 \quad \text{if } a_\nu = 0 .
\end{aligned}
$$

They effect that the correspondent binomial coefficients

$$\binom{k_{\nu'}}{k_{\nu'} - 1} \quad \text{or} \quad \binom{k_\nu + 1}{k_{\nu-1}}$$

drop out becoming 1, and that the linear form $S_0(\mathfrak{M})$ is changed to no longer homogeneous linear functions $S_\nu(\mathfrak{M})$ of the remaining summation variables.

We illustrate this in the two cases where all but one or all of the coefficients $a_1, \cdots, a_{n-2}$ are specialized to 0.

(i) $\underline{a_1, \cdots, a_{r-1},\ a_{r+1}, \cdots, a_{n-2} = 0;\ a_r = a \neq 0}$

$\underline{(1 \le r \le n-2)}$

$$
(11.10)\quad \begin{cases}
p_k^{(\nu)} = \displaystyle\sum_{k',k''} \binom{k''}{k'}\binom{k - S_\nu(k',k'')}{k''} t^{k-\left[\frac{k}{n}\right]-S_\nu(k',k'')} a^{k-S'_\nu(k',k'')}, & \begin{pmatrix} k \ge 0 \\ \nu = 0, \cdots, r-1 \end{pmatrix} \\[2ex]
p_k^{(\nu)} = \displaystyle\sum_{k',k''} \binom{k''+1}{k'}\binom{k - S_\nu(k',k'')}{k''} t^{k-\left[\frac{k}{n}\right]-S_\nu(k',k'')} a^{k+1-S'_\nu(k',k'')}, & \begin{pmatrix} k \ge 0 \\ \nu = r, \cdots, n-2 \end{pmatrix} \\[2ex]
p_k^{(n-1)} = \displaystyle\sum_{k',k''} \binom{k''}{k'}\binom{k + 1 - S'_\nu(k',k'')}{k''} t^{k-\left[\frac{k}{n}\right]-S'_\nu(k',k'')} a^{k+1-S'_\nu(k',k'')} & \begin{pmatrix} k = 0 \\ \nu = n-1 \end{pmatrix}
\end{cases}
$$

with the linear functions

$$
S_\nu(k',k'') = rk' + (n-1-r)k'' + \begin{Bmatrix} \nu & \text{for } \nu = 0, \cdots, r-1 \\ \nu - r & \text{for } \nu = r, \cdots, n-2 \\ 0 & \text{for } \nu = n-1 \end{Bmatrix}
$$

and

$$
S_\nu(k',k'') = S_\nu(k',k'') + k' = (r+1)k' + (n-1-r)k'' + \begin{Bmatrix} \nu & \text{for } \nu = 0, \cdots, r-1 \\ \nu - r & \text{for } \nu = r, \cdots, n-2 \\ 0 & \text{for } \nu = n-1 \end{Bmatrix} .
$$

(ii) $\underline{a_1, \cdots, a_{n-2} = 0}$

$$(11.11) \qquad p^{(\nu)} = \sum_{k'} \binom{k + e_{n-1}^{(\nu)} - S_\nu(k')}{k'} t^{k-\left[\frac{k}{n}\right]-S_\nu(k')} a^{k+e_{n-1}^{(\nu)}-S'_\nu(k')} \qquad \left(\begin{array}{l} k = 0 \\ \nu = n-1 \end{array}\right),$$

with the linear functions

$$S_\nu(k') = \begin{Bmatrix} (n-1)k' + \nu & \text{for } \nu = 0, \cdots, n-2 \\ (n-1)k' & \text{for } \nu = n-1 \end{Bmatrix}$$

and

$$S'_\nu(k') = S_\nu(k') + k' = \begin{Bmatrix} nk' + \nu & \text{for } \nu = 0, \cdots, n-2 \\ nk' & \text{for } \nu = n-1 \end{Bmatrix} .$$

We illustrate (11.10) and (11.11) by the lowest case:

<u>$n = 3$</u>

In (11.10) for $n = 3$ the only possibility is $r = 1$. But then $a_0 = 1$; $a_1, a_2 = a \neq 0$, and <u>no</u> coefficient is specialized to 0. Hence formulae (11.10) must coincide with (11.9), which is confirmed at once.

Formulae (11.11) for $n = 3$ specialize to

$$(11.12) \quad \left\{ \begin{aligned} p_\kappa^{(0)} &= \sum_{k'} \binom{k - 2k'}{k'} t^{k-\left[\frac{k}{3}\right]-2k'} a^{k-3k'} \\ p_\kappa^{(1)} &= \sum_{k'} \binom{k - 2k' - 1}{k'} t^{k-\left[\frac{k}{3}\right]-2k'-1} a^{k-3k'-1} \\ p_\kappa^{(2)} &= \sum_{k'} \binom{k + 1 - 2k'}{k'} t^{k-\left[\frac{k}{3}\right]-2k'} a^{k+1-3k'} \end{aligned} \right. \qquad (k \geq 0) .$$

The term with $k = 0$, $k' = 0$ in the second formula is an example for the necessity of our deviating convention after (10.2), (10.2') about the binomial coefficients with negative "numerator." From the recurrency

$$p_0^{(1)} = 1p_{-3}^{(1)} + 0p_{-2}^{(1)} + 1p_{-1}^{(1)}$$

with

$$p_{-3}^{(1)} = 0, \quad p_{-2}^{(1)} = 1, \quad p_{-1}^{(1)} = 0 \; ,$$

it is obvious that $p_0^{(1)} = 0$. But (11.12) would yield a non-zero value $p_0^{(1)}$, with negative exponents of t and a into the bargain, if the binomial coefficient $\binom{-1}{0}$ of the first term of the sum would be given the usual value 1.

b) Cases with $a_{n-1} = 0$

As we saw in Section 10, in these cases, the general reduction (10.4) to unrestricted summation is ineffective, and we can achieve our aim in the same way only if there is at least one consecutive pair of recurrency coefficients $a_{\nu'}$, $a_{\nu'+1}$ with $0 \leq \nu' \leq n - 2$, which are not specialized to 0.

We shall consider here again only cases where all but one of the coefficients $a_1, \cdots, a_{n-2}$ are specialized to 0; in the case where all of them are 0, the recurrency

$$p_k^{(\nu)} = p_{k-n}^{(\nu)}$$

is trivial.

Let $a_r = a \neq 0$ be the only coefficient remaining intact. For $r = 1$ the pair $a_0 = 1$, $a_r = a$ satisfies the above condition, for $r = 2, \cdots, n-2$ however it is not satisfied. In both cases, we have to go back to our general result (8.8).

(i) $a_1 = a \neq 0;\; a_2, \cdots, a_{n-1} = 0$

Here, in (8.8) are to be added the summation conditions

$$K_2 = \cdots = K_{n-2} = K \quad ,$$

so that now

$$S(\mathfrak{M}) = S(K_0, K) = K_0 + (n - 1)K \; .$$

Thus (8.8) becomes

$$
\begin{aligned}
p_k^{(0)} &= \sum_{S(K_0,K)=k+n} \binom{K}{K_0} \frac{K_0}{K}\, t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&= \sum_{S(K_0,K)=k+n} \binom{K-1}{K_0-1} t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&= \sum_{S(K_0,K)=k} \binom{K}{K_0} t^{K-\left[\frac{k}{n}\right]} a^{K-K_0} \qquad \begin{pmatrix} k \geq 0 \\ \nu = 0 \end{pmatrix} \qquad (11.13)\\
p_k^{(\nu)} &= \sum_{S(K_0,K)=k+(n-\nu)} \binom{K}{K_0} t^{K-\left[\frac{k}{n}\right]-1} a^{K-K_0} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 1,\cdots,n-1 \end{pmatrix}.
\end{aligned}
$$

Since in the summation condition K_0 has coefficient 1, it can be eliminated, putting

$$
K_0 = \left\{ \begin{matrix} k - (n-1)K & \text{for } \nu = 0 \\ k + (n-\nu) - (n-1)K & \text{for } \nu = 1,\cdots,n-1 \end{matrix} \right\} .
$$

Making this substitution, we can however no longer silently pass over the summation conditions $0 \leq K_0 \leq K$. Thus we obtain

$$
(11.14) \quad \left\{
\begin{aligned}
p_k^{(0)} &= \sum_{(n-1)K \leq k \leq nK} \binom{K}{k-(n-1)K} t^{K-\left[\frac{k}{n}\right]} a^{nK-k} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 0 \end{pmatrix} \\
p_k^{(\nu)} &= \sum_{(n-1)K \leq k+(n-\nu) \leq nK} \binom{K}{k+(n-\nu)-(n-1)K} \times \\
&\quad \times t^{K-\left[\frac{k}{n}\right]-1} a^{nK-k-(n-\nu)} \\
&\qquad \begin{pmatrix} k \geq 0 \\ \nu = 1,\cdots,n-1 \end{pmatrix} .
\end{aligned}
\right.
$$

We illustrate this by the lowest case:

$\underline{n = 3}$

$$
(11.15)\quad \begin{cases}
p_k^{(0)} = \displaystyle\sum_{2K \le k \le 3k} \binom{K}{k - 2K} t^{K - \left[\frac{k}{3}\right]} a^{3K-k} \\[2ex]
p_k^{(1)} = \displaystyle\sum_{2K \le k+2 \le 3K} \binom{K}{k + 2 - 2K} t^{K - \left[\frac{k}{3}\right] - 1} a^{3K-k-2} \\[2ex]
p_k^{(2)} = \displaystyle\sum_{2K \le k+1 \le 3K} \binom{K}{k + 1 - 2K} t^{K - \left[\frac{k}{3}\right] - 1} a^{3K-k-1} \qquad (k \ge 0)
\end{cases}
$$

Formulae (11.9), (11.12), (11.15) together cover all possible cases of generalized Fibonacci numbers of order $n = 3$ with time impulses.

(ii) $a_1, \cdots, a_{n-1},\ a_{r+1}, \cdots, a_{n-1} = 0;\ a_r = a \ne 0$
$\underline{(2 \le r \le n - 2)}$

Here, in (8.8) are to be added the summation conditions

$$
K_0 = \cdots = K_{r-1} = K, \qquad K_r = \cdots = K_{n-1} = K' ,
$$

so that now

$$
S(\mathfrak{M}) = S(K, K') = rK + (n - r)K' .
$$

Thus (8.8) becomes

$$
(11.16)\quad \begin{cases}
p_k^{(\nu)} = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K'}{K} \frac{K}{K'}\, t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \\[2ex]
\qquad = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K' - 1}{K - 1} t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \\[2ex]
\qquad = \displaystyle\sum_{S(K,K')=k-\nu} \binom{K'}{K} t^{K' - \left[\frac{k}{n}\right]} a^{K'-K} \qquad \left(\begin{matrix} k \ge 0 \\ \nu = 0, \cdots, r - 1 \end{matrix}\right) \\[2ex]
p_k^{(\nu)} = \displaystyle\sum_{S(K,K')=k+(n-\nu)} \binom{K'}{K} t^{K' - \left[\frac{k}{n}\right] - 1} a^{K'-K} \qquad \left(\begin{matrix} k \ge 0 \\ \nu = r, \cdots, n-1 \end{matrix}\right)
\end{cases}
$$

Since here in the summation condition, both variables K, K' have coefficients $r, n - r > 1$, neither of them can be eliminated, so that by (11.16), other than (11.13), has to be considered as the final result.

There is, however, one very special case in which a different possibility of achieving unrestricted summation presents itself, viz., if both coefficients $r, n - r$ are equal, or else:

$$\underline{n = 2r}$$

In this case the summation restriction is

$$\frac{n}{2}(K + K') = \begin{cases} k - \nu & \text{for } \nu = 0, \cdots, n/2 - 1 \\ k + (n - \nu) & \text{for } \nu = n/2, \cdots, n - 1 \end{cases} .$$

Hence the sequences $p^{(\nu)}$ contain non-zero terms only for $k \equiv \nu \bmod n/2$, respectively. Putting accordingly

$$k = \begin{cases} \frac{n}{2}h + \nu & \text{for } \nu = 0, \cdots, \frac{n}{2} - 1 \\ \frac{n}{2}h + \left(\nu - \frac{n}{2}\right) & \text{for } \nu = \frac{n}{2}, \cdots, n - 1 \end{cases} \qquad (h \geq 0),$$

the restriction becomes

$$K + K' = \begin{cases} h & \text{for } \nu = 0, \cdots, \frac{n}{2} - 1 \\ h + 1 & \text{for } \nu = \frac{n}{2}, \cdots, n - 1 \end{cases} .$$

Here K', say, can be eliminated by the substitution

$$K' = \begin{cases} h - K & \text{for } \nu = 0, \cdots, \frac{n}{2} - 1 \\ h + 1 - K & \text{for } \nu = \frac{n}{2}, \cdots, n - 1 \end{cases} .$$

Thus in this very special case the non-zero terms of the sequences $p_k^{(\nu)}$ are the unrestricted sums

$$p^{(\nu)}_{\frac{n}{2}h+\nu} = \sum_K \binom{h-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 0,\cdots,\frac{n}{2}-1\end{matrix}\right)$$

(11.17)

$$p^{(\nu)}_{\frac{n}{2}h+\left(\nu-\frac{n}{2}\right)} = \sum_K \binom{h+1-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h+1-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = \frac{n}{2},\cdots,n-1\end{matrix}\right)$$

We illustrate this by the lowest case:

$$\underline{n = 4}$$

$$p^{(\nu)}_{2h+\nu} = \sum_K \binom{h-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 0,1\end{matrix}\right)$$

(11.18)

$$p^{(\nu)}_{2h+(\nu-2)} = \sum_K \binom{h+1-K}{K} t^{h-\left[\frac{h}{2}\right]-K} a^{h+1-2K} \qquad \left(\begin{matrix} h \geq 0 \\ \nu = 2,3\end{matrix}\right).$$

However formulae (11.17), (11.18) are immediate consequences of the general result (11.8) for $n = 2$, because considering only the non-zero terms, the corresponding recurrency formulae reduce to those for the generalized Fibonacci numbers of order $n = 2$ with time impulse. This shows the underlying true reason why reduction to unrestricted summation is possible in this very special case (and in similar cases with any proper division of n instead of 2 as well), in spite of what has been said in Section 10.

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