NUMBERS GENERATED BY THE FUNCTION $\exp(1-e^x)$

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A sequence of numbers $\{C_n,\ n = 0,1,2,\cdots\}$ is defined from its generating function $\exp(1 - e^x)$. A series representation for C_n (which is analogous to Dobinski's formula), a relationship with the Stirling numbers of the second kind, a recurrence relation between the C_n and a difference equation satisfied by C_n are obtained. The relationships between the Bell numbers and $\{C_n\}$ are also investigated. Finally, three determinantal representations for C_n are given. The 'Aitken Array' for C_n, $1 \le n \le 21$ is given in the appendix.

1. INTRODUCTION AND SUMMARY

While studying the moment properties of a discrete random variable associated with the Stirling numbers of the second kind, σ_n^j, we encountered an interesting sequence of numbers. More explicitly, let X be a discrete random variable with probability distribution

$$(1.1)\qquad P\{X = j\} = \sigma_n^j / B_n, \qquad j = 1,2,\cdots,n$$

where

$$\sum_{j=1}^{n} \sigma_n^j = B_n, \quad n = 1,2,\cdots$$

are called the Bell numbers. The k^{th} moment of the random variable X is given by

$$(1.2)\qquad D(X^k) = \sum_{j=1}^{n} j^k \sigma_n^j / B_n = B_n^{(k)} / B_n \quad \text{(say)};$$

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and the first six values of $B_n^{(k)}$ are given by

$$
\begin{aligned}
B_n^{(0)} &= B_n \\
B_n^{(1)} &= B_{n+1} - B_n \\
B_n^{(2)} &= B_{n+2} - 2B_{n+1} \\
B_n^{(3)} &= B_{n+3} - 3B_{n+2} + 0B_{n+1} + B_n \\
B_n^{(4)} &= B_{n+4} - 4B_{n+3} + 0B_{n+2} + 4B_{n+1} + B_n \\
B_n^{(5)} &= B_{n+5} - 5B_{n+4} + 0B_{n+3} + 10B_{n+2} + 5B_{n+1} - 2B_n \ .
\end{aligned}
\tag{1.3}
$$

This led us to look for an expression for $B_n^{(k)}$ in terms of the Bell numbers $B_{n+k}, B_{n+k-1}, \cdots, \cdots, B_n$ of the form

$$
B_n^{(k)} = \sum_{i=0}^{k} \binom{k}{i} C_i \, B_{n+k-1}
\tag{1.4}
$$

The first few C_i, $i = 1, 2, \cdots$ are given by $C_0 = 1$, $C_1 = -1$, $C_2 = 0$, $C_3 = 1$, $C_4 = 1$, $C_5 = -2$, $C_6 = -9$, $C_7 = -9$ and $C_8 = 50$. In this paper we will study some properties of the sequence $\{C_n\}$. In the next section, we give an ad hoc definition of $\{C_n\}$ in terms of the generating function $\exp(1 - e^x)$ and prove some properties. We also derive a relationship between Stirling numbers of the second kind and the C_n. In Section 3, we will derive some relationships between the Bell numbers and the C_n. In Section 4, we will obtain some determinantal representations for the C_n. The proofs are closely related to the proofs (due to several authors) in the case of Bell numbers as summarized by Finlayson in his thesis [1].

2. THE NUMBERS GENERATED BY THE FUNCTION $\exp(1 - e^x)$

Definition: The sequence $\{C_n, \ n = 0, 1, 2, \cdots\}$ is defined by its exponential generating function,

$$
\sum_{k=0}^{\infty} C_k \frac{x^k}{k!} = \exp(1 - e^x) \ .
\tag{2.1}
$$

From the power series expansion of $\exp(1 - e^x)$ we will give an infinite series representation for C_k.

Proposition 1:

$$C_k = e \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{r!}, \qquad k = 0, 1, 2, \cdots . \tag{2.2}$$

Proof: From the definition we note that C_k is the coefficient of $x^k/k!$ in the Maclaurin series expansion of $\exp(1 - e^x)$.

$$\begin{aligned} \exp(1 - e^x) &= e \sum_{r=0}^{\infty} (-1)^r e^{xr}/r! \\ &= e \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{k=0}^{\infty} \frac{x^k r^k}{k!} \\ &= e \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{k!} \end{aligned} ,$$

which shows that

$$C_k = e \sum_{r=0}^{\infty} (-1)^r \frac{r^k}{r!}, \qquad k = 0, 1, 2, \cdots .$$

We will use this series representation to obtain the relationship between the Stirling numbers of the second kind σ_k^j and C_k. We define $\sigma_0^0 = 1$ and $\sigma_k^0 = 0$, $k = 1, 2, \cdots$.

Proposition 2:

$$C_k = \sum_{j=1}^{k} (-1)^j \sigma_k^j \quad . \tag{2.3}$$

Proof. In terms of the j^{th} differences of powers of zero, $\Delta^j(0^k)$, we have, according to Jordan [3],

$$
\begin{aligned}
r^k &= \sum_{j=0}^{k} \binom{r}{j} \frac{\Delta^j(0^k)}{j!} \\
C_k &= r \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, r^k \\
&= e \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{j=0}^{k} \binom{r}{j} \Delta^j(0^k) \\
&= e \sum_{r=0}^{\infty} (-1)^r \sum_{j=0}^{k} \frac{\Delta^j(0^k)}{j!\,(r-j)!} \\
&= e \sum_{j=0}^{k} \frac{\Delta^j(0^k)}{j!} (-1)^j \sum_{r=j}^{\infty} \frac{(-1)^r(-1)^{-j}}{(r-j)!} \\
&= \sum_{j=0}^{k} (-1)^j \frac{\Delta^j(0^k)}{j!}
\end{aligned}
$$

which proves the result since $\Delta^j(0^k) = j!\sigma_k^j$.

Customarily, Stirling numbers of the first kind are defined as numbers with alternate signs, whereas Stirling numbers of the second kind are defined as numbers with positive signs. The relation (2.3) for the C_n, and the corresponding relation for the Bell numbers B_n, given by

$$B_n = \sum_{j=0}^{n} \sigma_n^j ,$$

suggest that the Stirling numbers of the second kind may also be defined with alternate signs.

Using proposition 1, we now obtain a recursive relation between the C-numbers.

Proposition 3.

$$C_{k+1} = -\sum_{j=0}^{k} \binom{k}{j} C_j \qquad k = 0,1,\cdots; \; C_0 = 1 . \tag{2.4}$$

Proof:

$$\begin{aligned} C_{k+1} &= e \sum_{r=1}^{\infty} (-1)^r \frac{r^{k+1}}{r!} \\ &= e \sum_{s=0}^{\infty} (-1)^{s+1} \frac{(s+1)^k}{s!} \\ &= e \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!} \sum_{j=0}^{k} \binom{k}{j} s^j \\ &= -\sum_{j=0}^{k} \binom{k}{j} e \sum_{s=0}^{\infty} \frac{(-1)^s s^j}{j!} = -\sum_{j=0}^{k} \binom{k}{j} C_j . \end{aligned}$$

In the next proposition we will show that C_n satisfies an n^{th} order difference equation. As before, let Δ denote the difference operator and let $E \equiv 1 + \Delta$, so that $E^j C_0 = C_j$, $j = 1,2,\cdots$.

Proposition 4:

$$\Delta^n C_1 = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} C_{j+1} = -C_n, \qquad n = 1,2,\cdots . \tag{2.5}$$

Proof. The first equality will be established by the binomial expansion of $(E - 1)^n$, and the second equality follows from proposition 1. For completeness, the proof is sketched on the following page.

$$\Delta^n C_1 = (E - 1)^n E C_0 = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} E^j E C_0$$

$$= e \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}\sum_{r=0}^{\infty} (-1)^r \frac{r^{j+1}}{r!}, \quad \text{from (2.2)}$$

$$= e \sum_{r=1}^{\infty} \frac{(-1)^r r}{r!} \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} r^j$$

$$= -e \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!} (r - 1)^n = -C_n .$$

The difference equation $\Delta^n C_1 = -C_n$ can be used on computing $C_1, C_2, \cdots, C_n$ for small values of n. This computation can be arranged in a triangular array

$$\begin{array}{llllll} C_1 & \Delta C_1 & \Delta^2 C_1 & \Delta^3 C_1 & \Delta^4 C_1 & \cdots \\ C_2 & \Delta C_2 & \Delta^2 C_2 & \Delta^3 C_2 & \cdots & \\ C_3 & \Delta C_3 & \Delta^2 C_3 & \cdots & & \\ C_4 & \Delta C_4 & \cdots & & & \\ C_5 & \cdots & & & & \end{array} \tag{2.6}$$

The first column gives us the value of C_n, $n = 1, 2, 3, \cdots$, the second column gives us the first differences, and the j^{th} column gives us the j^{th} differences of C_n, $n = 1, 2, 3, \cdots$. This table can be filled up as follows: Let us assume that we know $C_1 = -1$. Equation (2.5) for $n = 1$, with $\Delta C_1 = -C_1$ enables us to find $C_2 = C_1 + \Delta C_1 = 0$. Now using (2.5) again for $n = 2$, we find $\Delta^2 C_1 = -C_2 = 0$. Since $\Delta^2 C_1 + \Delta C_1 = \Delta C_2$ we find $\Delta C_2 = 1$ and since $\Delta^2 C_2 + C_2 = C_3$, we find $C_3 = 1$. Now using (2.5) again for $n = 3$, with $\Delta^3 C_1 = -C_3$, we find $\Delta^3 C_1 = -1$, and so on. A part of the difference array is as follows:

$$\begin{array}{rrrrrr} -1 & 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & -1 & -2 & 1 & \\ 1 & 0 & -3 & -1 & & \\ 1 & -3 & -4 & & & \\ -2 & -7 & & & & \\ -9 & & & & & \end{array} \tag{2.7}$$

The corresponding table for the Bell numbers B_n and their differences, based on $\Delta^n B_1 = B_n$ is given in Table 1 of Finlayson [1]. He used the same method of construction, which is at times referred to as the Aitken array by Moser and Wyman [4]. In the appendix we give the Aitken array for the C_n for $1 \leq n \leq 21$.

3. RELATIONSHIPS BETWEEN THE BELL NUMBERS B_n, AND THE C_n

It is well known (Riordan [5]) that the exponential generating function of the Bell numbers B_n is given by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \exp(e^x - 1) . \tag{3.1}$$

Since the generating functions of

$$b_n = \frac{B_n}{n!} \quad \text{and} \quad c_n = \frac{C_n}{n!}$$

are reciprocals of each other, following Riordan [5] we could have defined the sequence $\{c_n\}$ as the inverse sequence of $\{b_n\}$. From this property we can easily derive the following

Proposition 5:

$$\sum_{k=0}^{n} \binom{n}{k} B_k C_{n-k} = 0, \quad n = 1, 2, \cdots, \quad \text{with} \quad B_0 = C_0 = 1 . \tag{3.2}$$

A less obvious relationship between B_n and C_n is given by the following:

Proposition 6:

$$\sum_{j=0}^{n} \binom{n}{j} C_j B_{n+1-j} = 1, \qquad n = 0, 1, 2, \cdots \tag{3.3}$$

Proof: Differentiating (3.1) with respect to x, we obtain

$$\sum_{k=1}^{\infty} B_k \frac{x^{k-1}}{(k-1)!} = e^x \exp(e^x - 1) .$$

Multiplying this by the exponential generating function of C_n we obtain

$$\left(\sum_{j=0}^{\infty} C_j \frac{x^j}{j!}\right)\left(\sum_{k=1}^{\infty} B_k \frac{x^{k-1}}{(k-1)!}\right) = e^x$$

which implies that

$$\sum_{n=0}^{\infty} B_1^{(n)} \frac{x^n}{n!} = e^x$$

where

$$B_1^{(n)} = \sum_{j=0}^{n} \binom{n}{j} C_j B_{n+1-j} ,$$

as defined in the introduction.

Now it follows that $B_1^{(n)} = 1$, $n = 0, 1, 2, \cdots$, since

$$e^x = \sum_{n=0}^{\infty} 1 \frac{x^n}{n!}$$

is the exponential generating function of the sequence with unity in every place. A 'dual' to proposition 6 can be stated as

Proposition 7:

$$\sum_{j=0}^{n} \binom{n}{j} B_j\, C_{n+1-j} = -1, \qquad n = 0, 1, 2, \cdots \tag{3.4}$$

Proof. This follows along the same lines as that of Proposition 6, where we now differentiate the exponential generating function of the C_n.

4. DETERMINANTAL REPRESENTATIONS OF C_n

We noted in Section 3 that the sequences

$$\{b_n\} = \left\{\frac{B_n}{n!}\right\} \qquad \text{and} \qquad \{c_n\} = \left\{\frac{C_n}{n!}\right\}$$

are inverse sequences as defined on page 25 of Riordan [5]. On page 45, Riordan gives as a problem the representation of n^{th} number of the sequence $\{a'_n\}$ as a determinant of the elements of the inverse sequence $\{a_n\}$. This says

$$a'_n = (-1)^n a_0^{-n-1} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & \cdots & \\ a_3 & a_2 & a_1 & \cdots & \\ \vdots & \vdots & \vdots & a_0 & \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix} = (-1)^n a_0^{-n-1}\,\delta_n \quad \text{(say)}.$$

The following recursive relation for δ_n can be shown,

$$\delta_n = \sum_{k=0}^{n-1} (-1)^k a_0^k\, a_{k+1}\ {}_{n-k-1}, \quad \delta_0 = 1 .$$

Applying this result for the Bell numbers B_n, and C_n we will have

Proposition 8:

a)

(4.1)
$$\frac{C_n}{n!} = (-1)^n \begin{vmatrix} \frac{B_1}{1!} & B_0 & & & \\ \frac{B_2}{2!} & \frac{B_1}{1!} & B_0 & & \\ \frac{B_3}{3!} & \frac{B_2}{2!} & \frac{B_1}{1!} & & \\ \cdots & \cdots & \cdots & \cdots & B_0 \\ \frac{B_n}{n!} & \frac{B_{n-1}}{(n-1)!} & \frac{B_{n-2}}{(n-2)!} & & \frac{B_1}{1!} \end{vmatrix} = (-1)^n \xi_n \quad \text{(say)}$$

(4.2) b)
$$(-1)^n \frac{C_n}{n!} = \sum_{k=0}^{n-1} (-1)^k \frac{B_{k+1}}{(k+1)!} \xi_{n-k-1} \quad .$$

In Proposition 3, we have shown that

$$C_{n+1} = -\sum_{j=0}^{n} \binom{n}{j} C_j, \qquad n = 0,1,2,\cdots$$

with $C_0 = 1$. From this nonsingular system of equations, using Cramer's rule, we can derive the following:

Proposition 9:

(4.3)
$$C_{n+1} = (-1)^n \begin{vmatrix} 1 & 1 & & & & & 0 \\ 1 & 1 & 1 & & & & \\ 1 & 2 & 1 & 1 & & & \\ 1 & 3 & 3 & 1 & 1 & & \\ \vdots & & & & & & 1 \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdot & \cdot & \cdots & \binom{n}{n} \end{vmatrix}$$

The corresponding determinantal representation for the Bell numbers which seems to be due to Ginsburg [2], is also quoted by Finlayson [1]. Ginsburg [2] derived another determinantal expression for the Bell numbers (also quoted by Finlayson [1]) and the corresponding representation for the C-numbers is given by the following:

<u>Proposition 10:</u>

$$C_{n+1} = (-1)^{n+1} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \\ \frac{1}{1!} & 1 & 2 & 0 & 0 & \\ \frac{1}{2!} & \frac{1}{1!} & 1 & 3 & 0 & \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 & 4 \cdots & \\ \vdots & \vdots & \vdots & & & n \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & & \cdots & 1 \end{vmatrix}$$

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APPENDIX

AITKEN ARRAY FOR THE C NUMBERS

n	C_n	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	-1	1	-0	-1	-1	2	9	9
2	0	1	-1	-2	1	11	18	-41
3	1	0	-3	-1	12	29	-23	-358
4	1	-3	-4	11	41	6	-381	-1355
5	-2	-7	7	52	47	-375	-1736	-1265
6	-9	0	59	99	-328	-2111	-3001	21590
7	-9	59	158	-229	-2439	-5112	18589	177063
8	50	217	-71	-2668	-7551	13477	195652	671803
9	267	146	-2739	-10219	5926	209129	867455	-318740
10	413	-2593	-12958	-4293	215055	1076584	548715	-24897365
11	-2180	-15551	-17251	210762	1291639	1625299	-24348650	-194276517
12	-17731	-32802	193511	1502401	2916938	-22723351	-218625167	-691883220
13	-50533	160709	1695912	4419339	-19806413	-241348518	-910508387	2126876237
14	110176	1856621	6115251	-15387074	-261154931	-1151856905	1216367850	52384151835
15	1966797	7971872	-9271823	-276542005	-1413011836	64510945	53600519685	
16	9938669	-1299951	-285813828	-1689553841	-1348500891	53665030630		
17	8638718	-287113779	-1975367569	-3038054732	52316529739			
18	-278475061	-2262481448	-5013422401	49278475007				
19	-2540956509	-7275903849	44265052506					
20	-9816860358	36989148757						
21	27172288399							

n	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}
1	-50	-267	-413	2180	17731	50533	-110176	-1966797
2	-317	-680	1767	19911	68264	-59643	-2076973	-11905466
3	-997	1087	21578	88175	8621	-2136616	-13982439	-30482853
4	90	22765	109853	96796	-2127995	-16119055	-44465292	220776103
5	22855	132618	206549	-2031199	-18247050	-60584347	176310811	3561302972
6	155473	339267	-1824550	-20278249	-78831397	115726464	3737613783	25168346191
7	494740	-1485283	-22102799	-99109646	36895067	3853340247	28905959974	
8	-990543	-23588082	-121212445	-62214579	3890235314	32759300221		
9	-24578625	-144800527	-183427024	3828020735	36649535535			
10	-169379152	-328227551	3644593711	40477556270				
11	-497606703	3316366160	44122149981					
12	2818759457	47438516141						
13	50257275598							

n	Δ_{16}	Δ_{17}	Δ_{18}	Δ_{19}	Δ_{20}
1	-9938669	-8638718	278475061	2540956509	9816860358
2	-18577387	269836343	2819431570	12357816867	
3	251258956	3089267913	15177248437		
4	3340526869	18266516350			
5	21607043219				