ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-162 Proposed by David A. Klarner, University of Alberta, Edmonton Alberta, Canada.

Suppose $a_{ij} \geq 1$ for $i, j = 1, 2, \cdots$, show there exists an $x \geq 1$ such that

$$(-1)^n \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x^2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x^n \end{vmatrix} \leq 0$$

for all n.

H-163 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the following identities:

1. $$\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = 2^{2n} F_{n+1}^2 - 1$$

2. $$5 \sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} = 2^{2n} L_{n+1}^2 - 1 \;,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

H-164 Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Generalize H-127 and find a recurrence relation for the product $C_n = A_n(x)B_n(y)$, where A_n and B_n satisfy the general second-order recurrence equations:

$$A_{n+1}(x) = R(x)A_n(x) + S(x)A_{n-1}(x) \tag{1}$$

$$B_{n+1}(y) = P(y)B_n(y) + Q(y)B_{n-1}(y), \tag{2}$$

$$n \geq 1 \quad \text{and} \quad A_0, A_1, B_0, B_1 \text{ arbitrary}.$$

H-165 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the identity

$$\sum_{i=1}^{n} \binom{n}{i} \frac{F_{ki}}{F_{k-2}^{i}} = \left(\frac{F_k}{F_{k-2}}\right)^n F_{2n} \qquad (k \neq 2),$$

where F_i denotes the i^{th} Fibonacci number.

SOLUTIONS

A BASIS OF FACT?

H-132 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania.

Let

$$F_1 = 1, \; F_2 = 1, \; F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n > 0$$

define the Fibonacci sequence. Show that the Fibonacci sequence is <u>not</u> a basis of order k for any positive integer k; that is, show that <u>not</u> every positive integer can be represented as a sum of k Fibonacci numbers, where repetitions are allowed and k is a fixed positive integer.

Solution by the Proposer.

Assume $\{F_n\}_1^\infty$ is a basis of order k, where k is some fixed positive integer. Then, in particular, for given $n > 0$, any positive integer $r \le F_n$ would have a representation in the form

$$r = \sum_{i=1}^{k} F_{n_i}\ , \tag{1}$$

where $n_1 \le n_2 \le \cdots \le n_k$ and $n_k \le n$. But the maximum number of distinct integers which could be formed by the right-hand side of (1) is clearly $\le n^k$. Thus each of the F_n integers $1, 2, 3, \cdots, F_n$ would have to be expressed in a form capable of representing at most n^k distinct integers. Since, by choosing n large enough, we can make $F_n > n^k$, a contradiction is obtained for the value of k under consideration. [The inequality $F_n > n^k$ follows from the fact that F_n is approximately $a^n/\sqrt{5}$ for large n, where $a = (1 + \sqrt{5})/2$].

SUM SHINE

H-133 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Characterize the sequences

i. $$F_n = u_n + \sum_{j=1}^{n-2} u_j$$

ii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j$$

iii. $$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{j=1}^{m} \sum_{j=1}^{i} u_j\ ,$$

by finding starting values and recurrence relations. Generalize.

Solution by D. V. Jaiswal, Holkar Science College, Indore, India.

We shall first prove the iii part.

$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j$$

$$\therefore \quad F_{n-1} = u_{n-1} + \sum_{j=1}^{n-3} u_j + \sum_{i=1}^{n-5} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-7} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j$$

$$F_{n-2} = u_{n-2} + \sum_{j=1}^{n-4} u_j + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \quad .$$

Since $F_n - F_{n-1} - F_{n-2} = 0$, we have

$$0 = (u_n - u_{n-1} - u_{n-2}) + \left(u_{n-2} - \sum_{j=1}^{n-4} u_j\right)$$

$$+ \left(\sum_{j=1}^{n-4} u_j - \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j\right) + \left(\sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j - \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j\right).$$

Cancelling out the terms, we get

$$u_n = u_{n-1} + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \quad .$$

(ii) Proceeding as above, we shall get

$$u_n = u_{n-1} + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j \ .$$

(i) Proceeding as above we shall get

$$u_n = u_{n-1} + \sum_{j=1}^{n-4} u_j \ .$$

Generalization. If

$$F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \cdots +$$

$$+ \sum_{s=1}^{n-2r} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1} \sum_{j=1}^{i} u_j \qquad ,$$

(r summations)

then proceeding as above, we shall get

$$u_n = u_{n-1} + \sum_{s=1}^{n-2r-2} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1} \sum_{j=1}^{i} u_j$$

Editorial Note: Professor Hoggatt obtained the solutions:

i) $u_n = u(n;\ 2,2)$

ii) $u_n = u(n;\ 3,3)$

iii) $u_n = u(n;\ 4,4)$ where $u(n;\ p,q)$ represents the generalized Fibonacci number.

See V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, No. 4, pp. 277-289.

CIRCLE TO THE RIGHT

H-134 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Evaluate the circulants

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}, \quad \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & & \cdots & \cdots \\ L_{n+1} & L_{n+2k} & \cdots & L_n \end{vmatrix}$$

Solution by the Proposer.

We recall that

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_{m-1} & a_0 & \cdots & a_{m-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_0 \end{vmatrix} = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} a_s \omega^{rs} \qquad (\omega = e^{2\pi i/m}).$$

Hence if we put

$$\Delta_m(F) = \begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+2k} & F_{n+2k} & \cdots & F_n \end{vmatrix},$$

$$\Delta_m(L) = \begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix},$$

we have

$$\Delta_m(F) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} F_{n+sk}\omega^{rs}, \quad \Delta_m(L) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} L_{n+sk}\omega^{rs}.$$

Put

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n.$$

Then

$$\sum_{s=0}^{m-1} F_{n+sk}\omega^{rs} = \frac{1}{\alpha - \beta} \sum_{s=0}^{m-1} (\alpha^{n+sk} - \beta^{n+sk})\omega^{rs}$$

$$= \frac{1}{\alpha - \beta}\left\{\alpha^n \frac{1-\alpha^{mk}}{1-\omega^r\alpha^k} - \beta^n \frac{1-\beta^{mk}}{1-\omega^r\beta^k}\right\}$$

$$= \frac{\alpha^n(1-\alpha^{mk})(1-\omega^r\beta^k) - \beta^n(1-\beta^{mk})(1-\omega^r\alpha^k)}{(\alpha-\beta)(1-\omega^r\alpha)(1-\omega^r\beta)}$$

$$= \frac{\alpha^n - \beta^n - (\alpha^{n+mk} - \beta^{n+mk}) - \omega^r(\alpha^n\beta^k - \alpha^k\beta^n - \alpha^{n+mk}\beta^k + \beta^{n+mk}\alpha^k)}{(\alpha-\beta)(1-\omega^r\alpha)(1-\omega^r\beta)}$$

$$= \frac{F_n - F_{n+mk} - (-1)^k\omega^r(F_{n-k} - F_{n+(m-1)k})}{(1-\omega^r\alpha^k)(1-\omega^r\beta^k)}$$

so that

$$\prod_{r=0}^{m-1}\sum_{s=0}^{m-1} F_{n+sk}\omega^{rs} = \prod_{r=0}^{m-1} \frac{F_n - F_{n+mk} - (-1)^k\omega^r(F_{n-k} - F_{n+(m-1)k})}{(1 - \omega^r\alpha^k)(1 - \omega^r\beta^k)}$$

$$= \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{(1 - \alpha^{mk})(1 - \beta^{mk})}$$

Therefore

$$(\star) \qquad \Delta_m(F) = \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}} .$$

Similarly,

$$\sum_{s=0}^{m-1} L_{n+sk}\omega^{rs} = \sum_{s=0}^{m-1} (\alpha^{n+sk} + \beta^{n+sk})\omega^{rs}$$

$$= \alpha^n \frac{1 - \alpha^{mk}}{1 - \omega^r\alpha^k} + \beta^n \frac{1 - \beta^{mk}}{1 - \omega^r\beta^k}$$

$$= \frac{\alpha^n + \beta^n - \alpha^{n+mk} - \beta^{n+mk} - \omega^r(\alpha^n\beta^k + \beta^n\alpha^k - \alpha^{n+mk}\beta^k - \beta^{n+m}\alpha)}{(1 - \omega^r\alpha)(1 - \omega^r\beta)}$$

$$= \frac{L_n - L_{n+mk} - (-1)^k\omega^r(L_{n-k} - L_{n+(m-1)k})}{(1 - \omega^r\alpha^k)(1 - \omega^r\beta^k)} .$$

It follows that

$$(\star\star) \qquad \Delta_m(L) = \frac{(L_n - L_{n+mk})^m - (-1)^{mk}(L_{n-k} - L_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}} .$$

Also solved by D. Jaiswal (India).

THE GREATEST INTEGER!

H-135 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

PART I

Show that

$$j + 1 = \sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j-d}{d} 2^{j-2d}(-1)^d \quad ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

PART II

Show that

$$F_{(j+1)n} = F_n \sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j-d}{d} L_n^{j-2d} (-1)^{(n+1)d} \quad ,$$

where $j \geq 0$ and $[j/2]$ is the greatest integer not exceeding $j/2$.

Solution by the Proposer.

PART I

We have (see "A Generalization of the Connection between the Fibonacci Sequence and Pascal's Triangle," by Joseph A. Raab, this quarterly, Vol. 1, No. 3, October 1963, pp. 25-26) that

$$\sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j-d}{d} 2^{j-2d}(-1)^d = x_j$$

and $x_{j+2} = 2x_{j+1} - x_j$ for all $j \geq 0$. Let S be the set of all integers $(j + 1) > 0$ for which the theorem is true, $1 = x_0$ and $2 = x_1$, so 1 and 2 are in S. Suppose q and $q + 1$ are in S, so that $q = x_{q-1}$ and $q + 1 = x_q$. Then

$$x_{q+1} = 2x_q - x_{q-1} = 2(q + 1) - q = q + 2 .$$

Thus $q + 2$ is in the set S and the proof is complete by mathematical induction.

PART II

The same reference as given in Part I yields the result that

$$\sum_{d=0}^{\left[\frac{j}{2}\right]} \binom{j - d}{d} L_n^{j-2d}(-1)^{(n+1)d} = x_j$$

and

$$x_{j+2} = L_n x_{j+1} + (-1)^{n+1} x_j$$

for all $j \geq 0$. Let S be the set of all integers $(j + 1) > 0$ for which the theorem is true. $F_n = F_n x_0$ and $F_{2n} = F_n L_n = F_n x_1$, so 1 and 2 are in S. Suppose q and $q + 1$ are in S, so that $F_{qn} = F_n x_{q-1}$ and $F_{(q+1)n} = F_n x_q$. Then

$$F_n x_{q+1} = F_n L_n x_q + F_n (-1)^{n+1} x_{q-1} = L_n F_{(q+1n} + (-1)^{n+1} F_{qn} = F_{(q+2)n}$$

by a known identity (see "Some Fibonacci Results Using Fibonacci-Type Sequences," by I. Dale Ruggles, this quarterly, Vol. 1, No. 2, April, 1963, p. 77). Thus $q + 2$ is in the set S and the proof is complete by mathematical induction.

Also solved by B. King, L. Carlitz, D. Jaiswal (India), and D. Zeitlin.

SQUEEZE PLAY

H-136 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California, and D. A. Lind, University of Virginia, Charlottesville, Va.

Let $\{H_n\}$ be defined by $H_1 = p$, $H_2 = q$, $H_{n+2} = H_{n+1} + H_n$ $(n \geq 1)$ where p and q are non-negative integers. Sh there are integers N and

k such that $F_{n+k} < H_n \leq F_{n+k+1}$ for all $n > N$. Does the conclusion hold if p and q are allowed to be non-negative reals instead of integers?

Solution by Gerald A. Edgar, Student, University of California, Santa Barbara, California.

In order for the result to be true, we must have $p > 0$ or $q > 0$. Let

$$a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2 .$$

Define $f(n) = [an + 1/2]$, for n a positive integer, where $[x]$ is the greatest integer in x (thus $f(n)$ is the nearest integer to an). We now prove that $f(f(n)) = f(n) + n$. The definition of f gives

$$\text{(1)} \qquad an + \tfrac{1}{2} \geq f(n) > an - \tfrac{1}{2}$$

$$\text{(2)} \qquad af(n) + \tfrac{1}{2} \geq f(f(n)) > af(n) - \tfrac{1}{2} .$$

But (1) is the same as

$$\frac{f(n)}{a} + \frac{1}{2a} > n \geq \frac{f(n)}{a} - \frac{1}{2a} ,$$

or, since $(1/a) = a - 1$,

$$(a - 1)f(n) + (a - 1)/2 > n \geq (a - 1)f(n) - (a - 1)/2$$

or

$$\text{(3)} \qquad af(n) + \frac{a}{2} - \frac{1}{2} > n + f(n) \geq af(n) - \frac{a}{2} + \frac{1}{2} .$$

Equations (2) and (3) give

$$\frac{a}{2} > f(n) + n - f(f(n)) \geq -\frac{a}{2} .$$

But $a/2 < 1$, and $f(n) + n - f(f(n))$ is an integer, so it must be zero, and we have

$$f(f(n)) = f(n) + n . \tag{4}$$

Because of its recurrence, H_n must have the form $H_n = ca^n + db^n$ for some constants c and d. Now $|b| < 1$, so

$$\lim_{n\to\infty} (aH_n - H_{n+1}) = \lim_{n\to\infty} (ca^{n+1} + dab^n - ca^{n+1} - db^{n+1})$$
$$= \lim_{n\to\infty} \sqrt{5}\, db^n = 0$$

Thus there is an integer N such that $|aH_n - H_{n+1}| < \frac{1}{2}$ for all $n \geq N$. In particular, $|aH_N - H_{N+1}| < \frac{1}{2}$, so, since H_{N+1} is an integer,

$$H_{N+1} = [aH_N + \tfrac{1}{2}] = f(H_N) .$$

It is now an easy induction to show that

$$H_{N+m} = f^m(H_N) \tag{5}$$

for $m = 0, 1, 2, \cdots$, where f^m is the m^{th} iterate of f defined by

$$f^0(x) = x$$
$$f^{n+1}(x) = f(f^n(x)) .$$

(Note that in particular, $f(F_n) = F_{n+1}$ for $n = 2, 3, \cdots$ for the Fibonacci numbers.) Since H_N is a positive integer, there is an integer k such that

$$F_{N+k} < H_N \leq F_{N+k+1} . \tag{6}$$

We may then obtain by induction (using the fact that f is strictly increasing on the positive integers)

$$F_{n+k} < H_n \leq F_{n+k+1} ,$$

for all $n \geq N$.

The result does not hold for non-negative reals in general; take

$$p = a/\sqrt{5}, \quad q = a^2/\sqrt{5};$$

then $H_n > F_n$ when n is even and $H_n < F_n$ when n is odd.

Also solved by J. Desmond, A. Shannon, and M. Yoder.

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